A Real Space Method for Averaging Lemmas

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Abstract. We introduce a new method to prove averaging lemmas, *i.e.* prove a regularizing effect on the average in velocity of a solution to a kinetic equation. The method does not require the use of Fourier transform and the whole procedure is performed in the 'real space'. We are consequently able to improve the known result when the integrability of the solution (or the right hand side of the equation) is different in space and in velocity. We also present a few counterexamples to test the optimality of the new results.

Résumé. Nous présentons une nouvelle méthode pour obtenir des lemmes de moyenne, c'est-à-dire un effet régularisant sur les moyennes en vitesse d'une équation cinétique. Cette méthode ne fait pas appel à la transformée de Fourier et toute la preuve se fait dans l'espace réel. Par conséquent, nous sommes capables d'améliorer les résultats connus quand l'intégrabilité de la solution (ou du second membre de l'équation) est différente en espace et en vitesse. Nous donnons également quelques contre-exemples pour vérifier le caractère optimal des nouveaux résultats.

*partially supported by the projects PICS 1014 and HPRN-CT-2002-00282 and a MCyT grant

Key words. Regularizing effects, averaging lemmas, dispersion estimates, X-ray transform.
Mots-clés. Effets régularisant, lemmes de moyenne, estimations de dispersion, transformée aux rayons X.
Mathematics Subject Classification 35B65, 82C40, 47G10.

1 Introduction

1.1 Main results

We study the following stationary kinetic equation

$$v \cdot \nabla_x f(x,v) = \Delta^{\alpha/2}\, g(x,v), \qquad x \in \mathbb{R}^d,\ v \in \mathbb{R}^d,\, 0 \le \alpha < 1. \tag{1.1}$$

As a transport equation, (1.1) has typically no regularizing effects (although in some cases it does, see at the end of the paper). However in many applications, the important physical quantity is not f itself but some of its moments so that we are interested in the optimal regularity of a quantity like

$$\rho(x) = \int_{\mathbb{R}^d} f(x,v)\phi(v)\, dv, \qquad \phi \in C_c^\infty(\mathbb{R}^d) \text{ given.} \tag{1.2}$$

It is also possible to consider an average on the sphere, with the same gain in regularity,

$$\tilde{\rho}(x) = \int_{|v|=1} f(x,v)\phi(v)\, d\gamma(v), \qquad \phi \in C_c^\infty(S^{d-1}) \text{ given.} \tag{1.3}$$

It turns out that the average ρ is more regular than f (as long as $\alpha < 1$ of course) as it was first noticed in [16] in an L^2 framework. Since that paper numerous works have been devoted to proving the optimal regularity for the average. The study is motivated by a large class of kinetic equations where the non linear term may be controlled by some average of the solution and by kinetic formulations where the average is the only important quantity.

The gain in regularity depends on the smoothness of f and g themselves. In comparison with previous works, we will use different spaces in velocity and space (see a more detailed discussion after the presentation of the results).

Consequently the functions f and g, defined in the phase space, are assumed to be in the following spaces

$$\begin{aligned} f \ &\in W_v^{\beta,\, p_1}(\mathbb{R}^d,\ L_x^{p_2}(\mathbb{R}^d)), \qquad && \beta \geq 0, \\ g \ &\in W_v^{\gamma,\, q_1}(\mathbb{R}^d,\ L^{q_2}(\mathbb{R}^d)), \qquad && -\infty < \gamma < 1. \end{aligned} \tag{1.4}$$

We denote by $\dot{B}^{s,r}_{t,u}$ the space which is obtained by real interpolation of two Besov spaces $B^{s,r}_u$ much like the classical Besov spaces can be obtained by real interpolation of Sobolev spaces.
The first result which we prove is the following theorem

Theorem 1.1 *Let f and g satisfy* (1.1) *and* (1.4) *with* $1 < p_2, q_2 < \infty$, $1 \leq p_1 \leq \min(p_2,\ p_2^*)$ *and* $1 \leq q_1 \leq \min(q_2,\ q_2^*)$ *where for a general p, p^* is the dual exponent of p, and assume moreover that $\gamma - 1/q_1 < 0$. Then,*

$$\|\rho\|_{\dot{B}^{s,r}_{\infty,\infty}} \leq C \|f\|^{1-\theta}_{W_v^{\beta,p_1}(L_x^{p_2})} \times \|g\|^{\theta}_{W_v^{\gamma,q_1}(L_x^{q_2})},$$

with

$$\begin{aligned} &\frac{1}{r} = \frac{1-\theta}{p_2} + \frac{\theta}{q_2}, \quad s = (1-\alpha)\theta, \\ &\theta = \frac{1+\beta-1/p_1}{1+\beta-1/p_1-\gamma+1/q_1}. \end{aligned} \tag{1.5}$$

Remarks.
1. This theorem contains most of the previous results (in particular the ones in [11] and [19]). It extends naturally the result given in [19] for $\beta < 1/2$.
2. We do not know whether in this case the average belongs to the true Sobolev space $W^{s,r}$. This optimal space was obtained in [3] for the usual case ($p_1 = p_2$, $q_1 = q_2$ and $\beta = 0$). This is certainly true if $p_1 = p_2$ and $q_1 = q_2$ but some difficulties could arise when the exponents are different. In any case, the simple but rough method of interpolation which we choose here cannot do better than $B^{s,r}_{\infty,\infty}$.
3. We do not have any trouble with exponents p_1 or q_1 equal to 1, only with p_2 or q_2.
4. The gain of regularity depends only on the regularity and integrability in velocity. This corresponds to [31] where the average is obtained in a space weaker but with the same homogeneity as ours by Sobolev embedding.

However contrary to [31], we have a limitation on the exponent in velocity (see the section about optimality).

Since we work with different spaces in space and velocity, the order in which the norms are taken is very important. In (1.4) we take first the norm in x and then the norm in v. As $p_1 \leq p_2$ or $q_1 \leq q_2$, this is a stronger assumption than the contrary (the norm in v first). So a natural question is whether it is possible to invert the order of the spaces. We are able to give a full answer only in dimension two.

Proposition 1.1 *If $d = 2$, Let f and g satisfy* (1.1) *but assume g is like in* (1.4) *but f in $L^{p_2}_x(W^{\beta,p_1}_v)$ (respectively $g \in L^{q_2}_x(W^{\beta,q_1}_v)$ and f like in* (1.4)*) provided we still have $p_1 \leq p_2$ and moreover $p_2 \leq 2$ (resp. $q_1 \leq q_2$ and $q_2 \leq 2$) then*

$$\|\rho\|_{\dot{W}^{s,r}} \leq C \|f\|^{1-\theta}_{L^{p_2}_x(W^{\beta,p_1}_v)} \times \|g\|^{\theta}_{L^{q_2}_x(W^{\gamma,q_1}_v)},$$

with

$$\begin{aligned} &\frac{1}{r} = \frac{1-\theta}{p_2} + \frac{\theta}{q_2}, \quad s = (1-\alpha)\,\theta < (1-\alpha)\,\theta_0, \\ &\theta_0 = \frac{1+\beta-1/p_1}{1+\beta-1/p_1-\gamma+1/q_1}. \end{aligned} \tag{1.6}$$

Remark.
This result is optimal in the sense that the conclusion is false if $p_2 > 2$ or $q_2 > 2$. We cannot prove an equivalent in higher dimensions, but we can show that the limit on p_2 or q_2 is in general d^* with $1/d^* = 1 - 1/d$, see the discussion at the end of the proof of the proposition.

Theorem 1.1 exhibits a sort of saturation: The regularity of the average does not improve when p_1 grows beyond p_2. At this point, it is very interesting to invert the norms because that means we work in the strongest space. So let us assume now that f and g satisfy

$$\begin{aligned} &f \in L^{p_2}_x(\mathbb{R}^d,\ W^{\beta,\,p_1}_v(\mathbb{R}^d)), \qquad \beta \geq 0, \\ &g \in L^{q_2}(\mathbb{R}^d,\ W^{\gamma,\,q_1}_v(\mathbb{R}^d)), \qquad -\infty < \gamma < 1. \end{aligned} \tag{1.7}$$

With this new framework, we can prove (but for the moment only in dimension two) the

Theorem 1.2 *Take $d = 2$. Let f and g satisfy* (1.1) *and* (1.7) *with $1 < p_2, q_2 < 2$, $p_2 \leq p_1$ and $q_2 \leq q_1$ and assume moreover that either $\gamma \leq 0$ and $g(x,v)\,\phi(v)$ is even in v or that $\gamma \leq -1/2$. Then,*

$$\|\rho\|_{\dot W^{s,r}} \leq C\|f\|^{1-\theta}_{L^{p_2}_x(W^{\beta,p_1}_v)} \|g\|^{\theta}_{L^{q_2}_x(W^{\gamma,q_1}_v)},$$

with

$$\begin{aligned}
&\frac{1}{r} = \frac{1-s}{p_2} + \frac{s}{q_2}, \quad \forall\, s = (1-\alpha)\,\theta < (1-\alpha)\,\theta_0,\\
&\theta_0 = \frac{\beta+\theta_f}{1+\beta-\gamma+\theta_f-\theta_g}, \quad \theta_f = 1-\frac{1}{p_1}+\frac{1/p_1-1/p_2}{1/p_2-1/2}\ \max(0,\,2/p_2-3/2),\\
&\theta_g = 1-\frac{1}{q_1}+\frac{1/q_1-1/q_2}{1/q_2-1/2}\ \max(0,\,2/q_2-3/2).
\end{aligned} \tag{1.8}$$

Remarks.
1. This theorem says for instance that if f and g belong to $L^{4/3}_x(L^2_v)$ then the average "almost" belongs to $W^{1/2,4/3}$. Therefore, it is still possible to gain one half derivative even if the functions are not L^2 in space.
2. It is difficult to say if this result is optimal or not, whether in some cases one half derivative is gained even if $p_2 > 2$ or $q_2 > 2$ for instance. In fact the only sure indication which we have is one of the counterexamples of the next section namely the one showing that for f and g in $L^1_x(L^\infty_v)$, no derivative may be gained on ρ.
3. If $g(.,-v)\phi(-v) \neq g(.,v)\,\phi(v)$ and $\gamma \geq -1/2$, it is still possible to get a better result than the regularity given by Theorem 1.1. The idea is to interpolate between the case $\gamma = -1/2$ in this theorem and the result given by Theorem 1.1 for $\gamma = 3/4$.
4. This theorem is only an example of what can be done. It is of course possible to mix a regularity like (1.7) for f with one like (1.4) for g thus obtaining different formulas. The derivation of such new results should be straightforward given the estimates presented in the proofs.

Theorem 1.2 is limited to exponents p_1 and q_1 less than 2. When one of these exponents is larger than 2 then it is sometimes possible to get an even better result. The idea is then more a combination of a regularization effect and a dispersion result and it gives the following result

Theorem 1.3 *Take d=2. Let f and g satisfy* (1.1) *and* (1.7) *with* $1 < p_2,\, q_2 < 2$, $p_2 \le p_1$ *and* $q_2 \le q_1$ *and assume moreover that* $\gamma \le 0$. *Then,*

$$\|\rho\|_{\dot W^{s,r}} \le C \|f\|^{1-\theta}_{L^{p_2}_x(W^{\beta,p_1}_v)} \|g\|^{\theta}_{L^{q_2}_x(W^{\gamma,q_1}_v)},$$

with

$$\begin{aligned}
&\frac{1}{r} = \frac{1-s}{r_f} + \frac{s}{r_g}, \quad \forall\, s = (1-\alpha)\,\theta < (1-\alpha)\,\theta_0,\\
&\theta_0 = \frac{(\beta+\theta_f)\times(1-2/q_2+2/r_g) + (\gamma+\theta_g)\times(2/p_2-2/r_f)}{1+\beta-\gamma+\theta_f-\theta_g-2/r_f+2/p_2+2/r_g-2/q_2},\\
&\theta_f < \min\left(1/2, 2(1-1/p_2)\right), \quad \frac{1}{r_f} = \frac{1}{2} + \frac{1}{2p_1} + \frac{2}{p_1}\left|\frac{3}{4} - \frac{1}{p_2}\right|,\\
&\theta_g < \min\left(1/2, 2(1-1/q_2)\right), \quad \frac{1}{r_g} = \frac{1}{2} + \frac{1}{2q_1} + \frac{2}{q_1}\left|\frac{3}{4} - \frac{1}{q_2}\right|.
\end{aligned} \tag{1.9}$$

Remarks.
1. As before, this theorem is only one example of what could be proved, the number of combinations being now quite large.
2. A somewhat strange effect is that Theorem 1.3 does not always give a better result than Theorem 1.2. It is always better in terms of integrability but as far as the regularity (number of derivatives) is concerned, it is an improvement if and only if $\theta_g + \gamma > \theta_f + \beta$.
3. The typical conclusion of Theorem 1.3 is that if f and g belong to $L^{4/3}_x(L^\infty_v)$ then the average belongs to $H^{1/2}$ in dimension two.
4. The hypothesis $\gamma \le 0$ is almost certainly necessary. For instance without it, the denominator in the formula for θ_0 could vanish.
5. We do not understand for the moment why the evenness condition on $g\,\phi$ is not necessary here whereas we need it for Theorem 1.2 and consequently whether this theorem would still be true without it.

The paper is organized as follows. We comment on the theorems before concluding the first section. The second section is devoted to counterexamples. Theorem 1.1 is proved in the third section where notations and basic ideas are introduced. Theorems 1.2 and 1.3 are proved in the fourth section. We give a new direct proof of the classical L^2 result in a first appendix, the interest of this proof being that the orthogonality property at the core of the estimate is quite apparent. Finally we explain in the appendix how one may recover the hypoelliptic regularity of [4] within our framework.

1.2 Some applications and comments

The results presented here, although they are proved for Equation (1.1), are also valid for unstationary equations, with exactly the same proof (a dimension d for the unstationary case corresponding to a dimension $d+1$ for the stationary one)

$$\frac{\partial f}{\partial t}(t,x,v) + v\cdot\nabla_x f = \Delta^{\alpha/2}\, g(t,x,v), \qquad t\in\mathbb{R}^+,\ x,v\in\mathbb{R}^d,\ 0\le\alpha<1. \tag{1.10}$$

They even apply to the situation where the flux is not simply v, *i.e.*

$$\frac{\partial f}{\partial t}(t,x,v) + a(v)\cdot\nabla_x f = \Delta^{\alpha/2}\, g(t,x,v), \qquad t\in\mathbb{R}^+,\ x\in\mathbb{R}^d,\ v\in\mathbb{R}^{d'}, \tag{1.11}$$

provided the flux a satisfies a so-called strong non degeneracy condition which reads: for any $K>0$, there is a constant C such that for any $\xi\in\mathbb{R}^d$, $\tau\in\mathbb{R}$ with $|\xi|+|\tau|\le 1$

$$\text{meas}\,\{v\ s.t.\ |v|\le K,\ |a(v)\cdot\xi-\tau|\le\varepsilon\}\le C\epsilon. \tag{1.12}$$

Equation (1.11) is typical of kinetic formulations, of scalar conservation laws for instance. Those formulations were derived in [23] and in [7] for a more complicated situation. Kinetic formulations were also obtained for isentropic gas dynamics in [24] and more recently for Ginzburg-Landau models with line energies in [18] and then [29]. The typical example of an application of averaging lemmas to kinetic equations is probably [10].
We refer the reader to [26] for an introduction to scalar conservation laws and kinetic formulations. We nevertheless remark that it is not known whether averaging lemmas give the optimal regularity for the solution to such an equation. In fact, in dimension 1 (that would correspond to a two-dimensional case for the stationary model), they don't: BV regularity was proved by Oleĭnik [25] some fifty years ago. Good examples where a careful analysis can produce more precised results than averaging lemmas (although not exactly regularity) can be seen in [17] and [30]. For Ginzburg-Landau models, that seems to be also the case, see [1] for instance.

Averaging lemmas were first obtained in [16] for f and g in $L^2_{x,v}$ without any derivatives in velocity. It was soon noticed that one could take g in a negative Sobolev space and still get a result (see [14] or [15]). The optimal result for $f\in L^p_{x,v}$ and $g\in W^{\gamma,p}_v L^p_x$ was proved in [11] and slighty improved (to get the average in a true Sobolev space) in [3]. The method involves a constant use of Fourier transform, interpolation between the L^2 and the L^1 case through dyadic decomposition in the Fourier space and therefore it requires Hardy spaces. This result was shown to be optimal in the two notes [21] and [22] (see also [13]).

Other methods exist (besides the one presented here), for example in [27] and [28]. The one developed in [6] is quite simple but it still uses Fourier transform and Hardy spaces and it is only able to handle $f \in L^p_{x,v}$ with $g \in W^{\gamma,p}_v L^p_x$ and the same exponent p. However with the recent addition of hypoelliptic regularity on f, this method is able to work with $f \in W^{\beta,p}_v L^p_x$ and $\beta > 0$. Other possible methods include wavelets such as in [9].

The additional regularity of f in velocity (under the form of derivatives) was first used in [19]. Just about the same time, a somewhat similar result was derived in [31]. The author worked with bounds like (1.7) for functions f and g with the condition $p_2 = q_2$ and he obtained a bound for the average in a Sobolev space which we may also get by Sobolev embedding from ours theorems in many cases.

The motivation for this paper came from [19] and [31] and it was to try to recover the results of [31] but with the right space: The main drawback in [31] is indeed that it does not provide the right number of derivatives, the improvement in regularity on the average being at least in part only an improvement in integrability.

The results presented here answer partially to that problem. We are able to recover the results of [31] and in fact to extend them to obtain the right number of derivatives. That this cannot always be done is also a consequence of one of our counterexamples.

One of the main interests of our method is that it completely avoids the use of Fourier transform (or decomposition in wavelets). In this respect it relates to [5] where the authors do not use Fourier transform in both variables but only in space.

What our method clearly highlights are the deep connections between averaging lemmas and the X-ray transform which reads

$$T_x f = \int_{-\infty}^{\infty} f(x - vt)\, dt.$$

The boundedness of this operator from L^p_x to $L^q_x(L^r_v)$ is in particular investigated in [12] (see also [8] and [32]). And in some sense, this paper is all about the study of the boundedness of a similar operator from L^p to $W^{\alpha,q}_x(W^{\beta,r}_v)$ with the aim of having α as high as possible.

2 Some counterexamples

We want to explain here why Theorem 1.1 is essentially optimal and give the corresponding counterexamples. This is divided in two parts. The first one proves with the assumptions made in Theorem 1.1 there is no hope to obtain a better result. The next one shows that the limitation $p_1 \leq p_2$ or $q_1 \leq q_2$ cannot be removed, *i.e.* if p_1 or q_1 are larger then we gain nothing for the average.

Thoughout all this section, we take as an averaging function ϕ any smooth function compactly supported in the annulus $\{1/2 \leq |x| \leq 1\}$. We also take $\alpha = 0$.

2.1 Optimality of Theorem 1.1

This is the exact analogue in our more general situation of the two notes of P-L Lions, [21] and [22], which show that the usual averaging lemmas (with $p_1 = p_2$, $q_1 = q_2$ and $\beta = 0$) are optimal. We nevertheless give here the counterexamples for the sake of completeness.

They are given in dimension two for simplicity. We do it in two steps. For the first one consider two C^∞_c functions a and b and take

$$\begin{aligned} f_N(x,v) &= N^{\delta(1/p_1-\beta)} \times a(N\,x_1, x_2/N)\; b(N^\delta v_1),\\ g_N(x,v) &= N^{1-\delta+\delta/p_1-\delta\beta} \times \partial_1 a(N\,x_1, x_2/N)\; N^\delta v_1\, b(N^\delta v_1). \end{aligned} \tag{2.1}$$

We then simply choose δ such that g_N belongs to the space $W^{\gamma,q_1}_v(L^{q_2}_x)$ uniformly in N for every q_2, so

$$\delta = \frac{1}{1 - 1/p_1 + \beta + 1/q_1 - \gamma}.$$

Notice that if $\gamma < 0$, we also have to require that $wb(w)$ be the γ derivative of some function. Moreover, we have

$$v \cdot \nabla_x f_N = g_N + h_N,$$

with for any r

$$\|h_N\|_{L^1_v(W^{1,r}_x)} \leq C N^{-2\delta}.$$

Therefore the contribution from h_N to the regularity of the average is one full derivative and (from the point of view of counterexample) we may neglect this term.
To finish with this counterexample, it is enough to notice that for any $1 \leq r \leq \infty$

$$\|\rho_N\|_{\dot W^{s,r}} \geq N^{s-\delta(1-1/p_1+\beta)}.$$

Hence for this norm to be bounded uniformly in N, we need that

$$s \leq \delta(1-1/p_1+\beta) = \frac{1-1/p_1+\beta}{1-1/p_1+\beta+1/q_1-\gamma},$$

which is precisely the value given by Theorem 1.1. This counterexample also shows that, provided $p_1 \leq p_2$ and $q_1 \leq q_2$, the regularity gained by averaging does not depend on the integrability in x of either f or g.

Now we prove that the exponent r given by Theorem 1.1 is optimal. To do so we consider

$$\begin{aligned} f_N(x,v) &= N^{1/p_2+\delta(1/p_1-\beta)} \times a(N\,x_1,x_2)\; b(N^\delta v_1),\\ g_N(x,v) &= N^{1+1/p_2-\delta+\delta/p_1-\delta\beta} \times \partial_1 a(N\,x_1,x_2)\; N^\delta v_1\, b(N^\delta v_1). \end{aligned} \tag{2.2}$$

To bound uniformly g_N in the space given by (1.4) (f_N was correctly normalized), we need to take

$$\delta = \frac{1+1/p_2-1/q_2}{1-1/p_1+\beta+1/q_1-\gamma}$$

We again have

$$v\cdot\nabla_x f_N = g_N + h_N,$$

with h_N more regular than g_N and so negligible for our purpose. Finally

$$\|\rho_N\|_{W^{s,r}} \geq N^{s+1/p_2-1/r-\delta(1-1/p_1+\beta)}.$$

Since we already know that s is at most the value given by Theorem 1.1, we take that one and deduce that for ρ_N to be uniformly bounded, we need that

$$\frac{1}{r} = \frac{1}{p_2} - \frac{s}{p_2} + \frac{s}{q_2},$$

which is the value given by theorem 1.1. If we care only about local regularity then any $1/r$ larger than this will do of course.

2.2 The conditions $p_1 \le \min(p_2,\, p_2^*)$ or $q_1 \le \min(q_2,\, q_2^*)$

The strange and somewhat disappointing condition in Theorem 1.1 is the requirement that $p_1 \le \min(p_2,\, p_2^*)$ or $q_1 \le \min(q_2,\, q_2^*)$. From the point of view of homogeneity, L^∞ in v should be the same as $H^{1/2}$ (in two dimensions) and give the same regularity, hence the importance of counterexamples which illustrate this limitation.

We consider the following function g_N

$$g_N(x,v) = \sum_{i=1}^{N} \sum_{j=1}^{N} (-1)^i \, \mathbb{I}_{|x_1 - i/N| \le 1/N^2} \times \delta(x_2 = j/N) \times \Phi_N(v).$$

Instead of true dirac masses, we should take approximations of them in L^1 so that g_N belong to L^1_x. However to keep things as simple as possible, we will do just as if Dirac masses belong to L^1. Then, we obviously have

$$\|g_N\|_{L^1_x\, L^\infty_v} = N \times N \times N^{-2} \times \|\Phi_N\|_{L^\infty} \le 1.$$

The function Φ_N will be determined later on but with an L^∞ norm less than one.

Next we define f_N by means of g_N

$$f_N(x,v) = a(x) \times \int_0^\infty g_N(x - vt, v)\, dt,$$

with $a(x)$ a regular function with compact support and value 1 in the ball of radius 2. Therefore we have

$$v \cdot \nabla_x f_N = g_N + h_N,$$

with

$$h_N = (v \cdot \nabla_x\, a) \times \int_0^\infty g_N(x - vt, v)\, dt.$$

It is obvious that h_N is at least as regular as g_N and so

$$\|v \cdot \nabla_x f_N\|_{L^1_x\, L^\infty_v} \le C. \tag{2.3}$$

Now let us compute the $L^1_x\, L^\infty_v$ norm of f_N. Given x and v the value of f_N depends on the number of times the line issued from x, and with direction v, crosses one of the small segments of which g_N is composed. This almost

never happens. For instance, if Nx_2 is an integer and if v is along the x_1-axis, then f_N is the average of Dirac masses. This case is avoided by assuming that $\Phi((a,0)) = 0$, for any a and it ensures that f_N does not exhibit any Dirac mass itself.
However, it remains the other cases where for example $x_1 = i/N \pm 1/N^2$ for some i. Then if $|v_1| \leq 1/N^2$, $f(x,v)$ is of order N. Finally the norm of f_N may be estimated as

$$\|f_N\|_{L^1_x\, L^\infty_v} \leq C\,(1 + N \times N \times N^{-2}) \leq C. \tag{2.4}$$

For ρ_N those points of concentration of f_N do not have any importance. Indeed ρ_N is the average of f_N in v and if f_N is of order N at some points, it is only for values of v in an angular sector of size N^{-2}. Consequently, ρ_N is at most of order one. Then consider a segment with relative coordinates (a,b) (relative with respect to x), this segment is seen from x with an angular variation of

$$\max\left(\frac{1}{N^2 b},\ \frac{b}{N^2\, a^2}\right).$$

Hence for a given x which is typically at a distance $1/2N$ of the closest line $x_2 = j/N$, the measure of the set of velocities v, such that the corresponding line crosses at least one segment, is

$$\sum_{j=1}^N \left(j \times \frac{1}{N\, j} + \sum_{i=j}^N \frac{j/N}{N^2\, i^2/N^2}\right) \sim 1.$$

Note that this also justifies that a given line almost never intersects more than one segment.
Now of course there is the question of the alterning signs in g_N which could produce cancellations in ρ_N. This is where the definition of Φ_N, and the fact that it is L^∞ but not in any Sobolev space, plays a crucial role. Indeed let us choose a Φ_N such that ρ_N is indeed of order 1 at the point (1/2, 1/2) for instance. This is possible but only because we do not need any derivability on Φ_N.
Then notice that ρ is almost periodic of period $2/N$. If the segments in g_N where equidistributed in the whole space, it would be exactly periodic but as it is, some small perturbation has to be expected from the compact support in g_N. Because the derivative of ρ_N is obviously at most of order N, this means that ρ_N is of order one on a domain a measure of order one also.

To conlcude this counterexample, we remark that ρ_N changes sign if we add $1/N$ to x_1 due to the alterning signs in g_N. Therefore, the derivative of ρ_N is exactly of order N and

$$\|\rho_N\|_{W^{s,1}_{loc}} \sim N^s. \tag{2.5}$$

The combination of (2.3), (2.4) and (2.5) shows that, although f_N and g_N are uniformly bounded in $L^1_x L^\infty_v$, ρ_N is not uniformly bounded in any $W^{s,1}_{loc}$, $s > 0$.

We turn to the case of exponents $p \geq 2$. We use polar coordinates in x and v, hence $x = re^{i\theta}$ $v = e^{i\phi}$. We take

$$g_N(x,v) = e^{iN\theta}\mathbb{I}_{r\leq N} \times e^{-iN\phi},$$

such that

$$\|g_N\|_{L^q_x L^\infty_v} = N^{2/q}.$$

As in the previous case, we define f_N as

$$f_N(x,v) = \left(\int_0^\infty g(x-vt,v)\,dt\right) \times a(r/N),$$

for a a C^∞_c function. We obtain

$$\|v\cdot\nabla_x f_N\|_{L^q_x L^\infty_v} \sim N^{2/q}. \tag{2.6}$$

Given any $x = re^{i\theta}$, if we choose $v = e^{i(\theta+\pi)}$, then $f_N(x,v)$ is equal to N, so that

$$\|f_N\|_{L^p_x L^\infty_v} \sim N^{1+2/p}. \tag{2.7}$$

Now given x and assuming that v is not parallel to x, then there are cancellations in the integral defining f_N. As a matter of fact, the order of f_N is the typical length on which there cannot be any cancellation. It is easy to see that this length is N/r or N if $r \leq 1$. Therefore, given the oscillation in ρ_N coming from the $e^{iN\theta}$ in g_N

$$\|\rho_N\|_{W^{s,1}_{loc}} \sim N^{1+s}. \tag{2.8}$$

As previously, this norm has to be bounded by the norm of g_N to the power s times the norm of f_N to the power $1-s$. Estimates (2.6), (2.7) and (2.8) have as a consequence that s has to satisfy

$$1+s \leq \frac{2s}{q} + 1 - s + \frac{2}{p} - \frac{2s}{p},$$

or

$$s \leq \frac{1/p}{1 - 1/q + 1/p}.$$

This again corresponds to the result predicted by Theorem 1.1.

Before ending this subsection, we would like to point out that these two classes of counterexamples do not rigorously allow us to conlude that the conditions $p_1 \leq \min(p_2, p_2^*)$, or the same for q_i, are absolutely necessary. At least a counterexample with an exponent $p_2 < 2$ for f and an exponent $q_2 > 2$ for g (or the converse) is missing.

3 Proof of Theorem 1.1

3.1 The problem

The idea of the method is quite simple, we regularize the operator $v \cdot \nabla_x$ by adding λf (λ is a parameter of interpolation which will be chosen later in terms of f and g)

$$(\lambda + v \cdot \nabla_x)\, f(x,v) = \Delta_x^{\alpha/2} g(x,v) + \lambda f(x,v).$$

We denote by T the operator

$$Tf(x) = \int_0^\infty \int_{\mathbb{R}^d} f(x - vt,\, v)\, e^{-\lambda t}\, \phi(v)\, dv\, dt. \tag{3.1}$$

Consequently

$$\rho(x) = \int_{\mathbb{R}^d} f(x,v)\phi(v)\, dv = \lambda Tf + \Delta_x^{\alpha/2} Tg. \tag{3.2}$$

We study this operator T in the next subsection and conclude the proof of Theorem 1.1 in the last one.

3.2 Estimates on T

We begin with the simple case where we only have L^1 regularity in velocity. In this case T can at best exchange derivability in v for derivability in x, more precisely we have

Lemma 3.1 *$\forall\, 0 \le s < 1,\ T:\ W^{s,1}_v(\mathbb{R}^d,\ L^p_x(\mathbb{R}^d)) \longrightarrow \dot W^{s,p}(\mathbb{R}^d)$, with norm $C\lambda^{s-1}$, for every $1 \le p \le \infty$.*

Proof. It is a direct computation, once one has noticed that

$$\partial_{x_i} f(x-vt,v) = -\frac{1}{t}\partial_{v_i}(f(x-vt,\,v))) + \frac{1}{t}(\partial_{v_i} f)(x-vt,\,v).$$

First of all, simply by commuting the integrals, it is obvious that

$$\left\| \int_{\mathbb{R}^d} f(x-vt,\,v)\phi(v)\,dv \right\|_{L^p} \le C\|f\|_{L^1_v L^p_x},$$

where C does not depend on t. Then we also obtain from our remark that

$$\left\| \partial_{x_i} \int_{\mathbb{R}^d} f(x-vt,\,v)\phi(v)\,dv \right\|_{L^p} \le \frac{C}{t}\|f\|_{W^{1,1}_v L^p_x}.$$

By interpolation, we conclude that for any $s < 1$

$$\left\| \int_{\mathbb{R}^d} f(x-vt,\,v)\phi(v)\,dv \right\|_{\dot W^{s,p}} \le \frac{C}{t^s}\|f\|_{W^{s,1}_v L^p_x},$$

and by integrating in t against $e^{-\lambda t}$ we get the desired result.
Notice that, if we work with the average $\tilde\rho$ on the sphere as given by (1.3), we have to use a slighty more complicated relation, decomposing the i-th coordinate vector e_i

$$e_i = \alpha v + w, \quad \textit{with } w\cdot v = 0,$$

we obtain

$$\begin{aligned}\partial_{x_i} f(x-vt,\,v) &= \alpha v\cdot\nabla_x f(x-vt,v) + w\cdot\nabla_x f(x-vt,v)\\ &= -\alpha\partial_t f - \frac{1}{t}\,w\cdot\nabla_v(f(x-vt,v)) + \frac{1}{t}\,(w\cdot\nabla_v f)(x-vt,v).\end{aligned}$$

Since $w\cdot\nabla_v$ is a derivative on the sphere, this leads to the same estimate. □

With exactly the same idea, one obtains for negative derivatives,

Lemma 3.2 *$\forall\, s \le 0,\ T:\ W^{s,1}_v(\mathbb{R}^d,\ L^p_x(\mathbb{R}^d)) \longrightarrow \dot W^{s,p}(\mathbb{R}^d) + L^p(\mathbb{R}^d)$, with norms $(C\,\lambda^{s-1},\ C\,\lambda^{-1})$.*

Note that in fact one obtains one term in $\dot W^{s,p}$ with norm λ^{s-1} and another one in L^p with norm λ^{-1}, which is what we mean by the notation $(C\,\lambda^{s-1},\ C\,\lambda^{-1})$.

This has now to be combined with the case of L^2 regularity in velocity. Here because of the hypothesis of Theorem 1.1, we also work in L^2 in x with the following estimate

Lemma 3.3 $\forall\, 0\le s<1/2,\ T:\ H^s_v(L^2_x)\longrightarrow \dot H^{s+1/2}$*, with norm* $C\lambda^{s-1/2}$.

Proof. It is simpler to prove the corresponding estimate for the dual operator of T,

$$T^*h(x,v)=\int_0^\infty h(x+vt)e^{-\lambda t}\phi(v)\,dt. \tag{3.3}$$

It is then equivalent to prove the lemma and to show that T^* sends $\dot H^{-s-1/2}$ in $H^{-s}_v(L^2_x)$ or L^2_x in $H^{-s}_v(\dot H^{s+1/2})$ since T^* commutes with the derivation in x. Now since for any h

$$\Delta_x^{s/2}(h(x+vt))=\frac{1}{t^s}\Delta_v^{s/2}(h(x+vt)),$$

this is a consequence of the fact that the operator $\bar T^*$, defined as

$$\bar T^*h(x,v)=\int_0^\infty \frac{1}{t^s}h(x+vt)e^{-\lambda t}\phi(v)dt,$$

sends L^2 in $\dot H^{1/2}$ with norm $C\lambda^{s-1/2}$ provided that $s<1/2$. This operator is the dual of $\bar T$

$$\bar T f(x)=\int_0^\infty\int_{\mathbb{R}^d}\frac{1}{t^s}f(x-vt,\,v)e^{-\lambda t}\phi(v)\,dv\,dt.$$

We use a classical $\bar T\bar T^*$ argument, more precisely

$$\int_{\mathbb{R}^{2d}}\Delta_x^{1/4}\bar T^*h\cdot\Delta_x^{1/4}\bar T^*h\,dx\,dv=\int_{\mathbb{R}^d}\Delta_x^{1/2}\bar T\,\bar T^*h\cdot h(x)\,dx.$$

We then observe that

$$\begin{aligned}\bar T\,\bar T^*h(x)&=\int_0^\infty\int_0^\infty\int_{\mathbb{R}^d}\frac{1}{(ut)^s}h(x+(t-u)v)e^{-\lambda t-\lambda u}|\phi(v)|^2\,dv\,du\,dt\\&=2\int_0^\infty\int_0^t\frac{1}{(ut)^s}h(x+(t-u)v)e^{-\lambda t-\lambda u}|\phi(v)|^2\,dv\,du\,dt.\end{aligned}$$

Now

$$\begin{aligned}
\bar{T}\,\bar{T}^* h(x) &= \int_0^\infty \int_0^t \int_{\mathbb{R}^d} \frac{1}{t^s (t-\tau)^s} h(x+\tau v) e^{-2\lambda t+\lambda\tau} |\phi(v)|^2 \, dv\, d\tau\, dt \\
&= \int_0^\infty \int_0^t \int_0^\infty \int_{S^d} \frac{r^{d-1}}{t^s (t-\tau)^s} h(x+r\tau w) e^{-2\lambda t+\lambda\tau} |\phi(rw)|^2 \, dw\, dr\, d\tau\, dt \\
&= \int_0^\infty \int_0^\infty \int_{|y|\le rt} \frac{r^{d-2}}{t^s} h(x-y)\ |\phi(ry/|y|)|^2\ \frac{e^{-2\lambda t+\lambda |y|/r}}{(t-|y|/r)^s} \cdot \frac{dy}{|y|^{d-1}} dr\, dt.
\end{aligned}$$

Hence when derivating $\bar{T}\bar{T}^*$, we obtain exactly the structure of a Riesz transform. Therefore the operator $\bar{T}\bar{T}^*$ is continuous from L^2 to $\dot{H}^1$ with norm $C\lambda^{2s-1}$, which concludes the proof of the lemma. $\square$

By the same method, we have the corresponding result for negative derivatives in velocity.

Lemma 3.4 $\forall\ s \le 0,\ T:\ H^s_v(L^2_x) \longrightarrow \dot{H}^{s+1/2} + \dot{H}^{1/2}$, *with norm* $C\left(\lambda^{s-1/2}, \lambda^{-1/2}\right)$.

The same remark as for Lemma 3.2 also holds here: For an integer number of derivatives, we obtain a sum of two terms, one in $H^{s+1/2}$ and the other in $H^{1/2}$.

To obtain the behaviour of T on any space of the form $W^{s,p_1}_v(L^{p_2}_x)$, we only have to interpolate between Lemma 3.1 and Lemma 3.3. For any $1 < p_2 < 2$, we point out first that the proof of Lemma 3.1 also shows that T sends $W^{s,1}_v(\mathcal{H}^1_x)$ in $\Delta_x^{-s/2}\mathcal{H}^1$ with $\mathcal{H}^1$ the Hardy space; This would also be true with any Banach space whose norm is invariant by translation (*i.e.* the norm of $f(x+h)$ is equal to the norm of f). Then we interpolate between $W^{s,1}_v(\mathcal{H}^1_x)$ and $H^s_v L^2_x$ to obtain $W^{s,p_2}_v L^{p_2}_x$ whose image by T is in the interpolation of $\Delta_x^{-s/2}\mathcal{H}^1$ and $\dot{H}^{s+1/2}$, that is $\dot{W}^{1-1/p_2,p_2}$. Finally we interpolate between $W^{s,1}_v(L^{p_2}_x)$ and $W^{s,p_2}_v L^{p_2}_x$, which is the space $W^{s,p_1}_v L^{p_2}_x$ with its image in the interpolate between $\dot{W}^{s,p_2}$ and $\dot{W}^{1-1/p_2,p_2}$. Therefore we have the following proposition

Proposition 3.1 *For any* $1 \le p_1 \le \min(p_2,\, p_2^*)$, *for any* s *with* $s \le 1/p_1$, *we have for* $s \ge 0$

$$T:\ W^{s,p_1}_v(\mathbb{R}^d,\ L^{p_2}_x(\mathbb{R}^d)) \longrightarrow \dot{W}^{1+s-1/p_1,\,p_2}(\mathbb{R}^d),\ \textit{with norm}\ C\lambda^{s-1/p_1},$$

and for s any negative integer

$$T:\ W_v^{s,p_1}(L_x^{p_2}) \longrightarrow \dot W^{1+s-1/p_1,\,p_2}(\mathbb{R}^d) + \dot W^{1-1/p_1,\,p_2}(\mathbb{R}^d),$$
$$\textit{with norms } (C\,\lambda^{s-1/p_1},\ C\,\lambda^{-1/p_1}).$$

Again the notation with the parenthesis for the norms means that the norm of the term in $\dot W^{1+s-1/p_1,\,p_2}(\mathbb{R}^d)$ is less than $C\,\lambda^{s-1/p_1}$ and the norm of the other term less than $C\,\lambda^{-1/p_1}$.

3.3 Conclusion of the proof of Theorem 1.1

We are ready to prove Theorem 1.1. We do it first with the additional assumption that $\beta < 1/p_1$. Indeed with that we may apply Proposition 3.1 to both f and g.

For the moment we will consider only $\gamma \geq 0$ or negative integers for γ. If $\gamma < 0$, Proposition 3.1 gives us two different terms for $T_k g_k$ in L^{q_2} norm, one has $1-1/q_1+\gamma$ derivatives and the other $1-1/q_1$. The first one will give us the result stated in Theorem 1.1, the other one would give even more regularity. However, the corresponding Besov spaces are also the interpolates of order θ, between L^{p_2} and $\dot W^{1-\alpha,q_2}$. Since the second term leads to an interpolation between the same spaces but of higher order, it is also included in the same space as the first term. Hence in the following we will forget about this second term.

We have

$$\rho = \lambda\rho^1 + \rho^2 = \lambda T f + T g,$$

with by Proposition 3.1

$$\|\rho^1\|_{\dot W^{1+\beta-1/p_1,\,p_2}} \leq C\lambda^{\beta-1/p_1} \times \|f\|_{W_v^{\beta,p_1} L_x^{p_2}},$$
$$\|\rho^2\|_{\dot W^{1+\gamma-1/q_1-\alpha,\,q_2}} \leq C\lambda^{\gamma-1/q_1} \times \|g\|_{W_v^{\gamma,q_1} L_x^{q_2}}.$$

We then minimize in λ according to the K-method of real interpolation. We refer to [2] or [20] for more details on this method.

Let us define the following function K

$$K(t) = \inf_{\rho=a^1+a^2}\Big(\|a^1\|_{\dot W^{1+\beta-1/p_1,\,p_2}} + t\|a^2\|_{\dot W^{1+\gamma-1/q_1-\alpha,\,q_2}}\Big).$$

If $\sup_{t>0} t^{-\theta} K(t)$ is finite, then ρ belongs to the space $\dot{B}^{s,r}_{\infty,\infty}$ which is the interpolation of order (θ,∞) of the two spaces $\dot{W}^{1+\beta-1/p_1,p_2}$ and $\dot{W}^{1+\gamma-1/q_1-\alpha,q_2}$ (here θ, s and r are given the values of Theorem 1.1). Now for any t, minimizing in λ, we take

$$\lambda = t^{1/(1+\beta-1/p_1-\gamma+1/q_1)},$$

and we find indeed, taking $a^1=\rho^1$ and $a^2=\rho^2$

$$K(t) \leq t^{(1+\beta-1/p_1)/(1+\beta-1/p_1-\gamma+1/q_1)} \times \|f\|^{1-\theta}_{W^{\beta,p_1}_v L^{p_2}_x} \times \|g\|^{\theta}_{W^{\gamma,q_1}_v L^{q_2}_x}.$$

Of course the operator which to any couple $(f,\ g)$ associates $\lambda Tf + T(v\cdot\nabla_x f)$ is well defined and linear. We use it on the spaces $\{f\in W^{\beta,p_1}_v(L^{p_2}_x)\ s.t.\ v\cdot\nabla_x f\in W^{\gamma,q_1}_v(L^{q_2}_x)\}$. Hence by complex interpolation, if we have proved Theorem 1.1 for values of γ which are integers, we deduce the result for any value.

It only remains to indicate how we prove Theorem 1.1 for $\beta\geq 1/p_1$. Clearly if Proposition 3.1 were true for these values, we would be done since the previous argument of real interpolation would not pose any difficulty.

If one tries to prove any of the lemmas in the previous subsection for $\beta\geq 1/p_1$, the problem is that we do not have enough integrability in t. More precisely, we would have to integrate a term in t^{-k} with $k\geq 1$ which is not possible. However

$$\begin{aligned}
Tf &= \int_0^\infty \int_{\mathbb{R}^d} \partial_t(t) f(x-vt,\ v) e^{-\lambda t}\ \phi(v)\, dv\, dt \\
&= \int_0^\infty \int_{\mathbb{R}^d} f(x-vt,\ v)\ \lambda t e^{-\lambda t}\ \phi(v) + \int_0^\infty\int_{\mathbb{R}^d} v\cdot\nabla_x f(x-vt,\ v) t e^{-\lambda t}\ \phi(v) \\
&= \int_0^\infty \int_{\mathbb{R}^d} f(x-vt,\ v)\ \lambda t e^{-\lambda t}\ \phi(v) + \frac{1}{\lambda}\int_0^\infty \int_{\mathbb{R}^d} g(x-vt,\ v)\lambda t e^{-\lambda t}\ \phi(v).
\end{aligned}$$

The first term has the same homogeneity as Tf but with more integrability around the origin in t. The second term, once it is multiplied by λ behaves exactly like the usual Tg.

Therefore, repeating this simple trick as many times as necessary, we avoid any problem of integrability in t for Tf and we may consider β as large as we want.

Notice finally that this would not work for Tg because we have used that $v\cdot\nabla_x f = g$ and we do not have anything like that for g.

4 Proofs of Prop. 1.1, Theorems 1.2 and 1.3

For simplicity, we only consider in this section averaging on the sphere of the kind (1.3). The results trivially extend to any more general averaging like (1.2).
Indeed an average like (1.2) is itself the average of quantities like (1.3) (but taking the averages on spheres of different radius). One may obtain a bound on an average on a sphere on radius r from the bound on the average for a sphere of radius 1 by a simple scaling argument and so eventually a bound on quantity like (1.2).
Moreover, in (1.3), we will take $\phi = 1$. This only means that we redefine f and g as for instance $\tilde{f} = f(x, v) \times \phi(v)$.

4.1 Proof of Proposition 1.1

From the proof of Theorem 1.1, the only thing we have to do is to prove the equivalent of Lemmas 3.1 and 3.2 (or almost the equivalent since we are losing a bit here)

$$T \ : \ L^p_x(\mathbb{R}^2,\, W^{s,q}_v(\mathbb{R}^2)) \longrightarrow \dot{W}^{s,p}(\mathbb{R}^2), \quad \text{with norm } C\lambda^{s-1},$$

if $q > 1$, $s \geq 0$ and in $W^{s,p}$ with norm $\lambda^{s-1} + \lambda^{-1}$ if $s < 0$. Note that of course in Theorem 1.1, we could reach the case $q = 1$. The fact that we cannot the estimate for this critival case here is the reason why in the end, after interpolating, we do not have the critical order of derivative in Prop. 1.1.
We deal with the derivatives in velocity just as in Lemma 3.1. Next simply by making the changes of variables $\lambda t \to t$ and $x\lambda \to x$ we may take $\lambda = 1$. Therefore by interpolation, it is enough to prove that the operator which to any f associates

$$\int_0^\infty \int_{S^1} f(x - vt,\, v)\, \frac{e^{-t}}{t^s}\, dv\, dt,$$

is continuous from $L^2_x(L^q_v)$ in L^2 if $s < 1$ and $q > 1$. By duality, we need to prove that the operator T^*_s which to any function $h(x)$ associates

$$T^*_s\, h = \int_0^\infty h(x + vt)\, \frac{e^{-t}}{t^s}\, dt, \tag{4.1}$$

is continuous from L^2_x in $L^2_x(\mathbb{R}^2,\, L^p(S^1))$ for any $2 \leq p < \infty$.

But now this is a consequence of the estimate that we already proved on T_s^*. Indeed we showed that T_s^* sends L^2 into $H_x^{1/2}L_v^2$. Now looking at formula (4.1), it is obvious that this implies that T_s^* sends L^2 into $H_v^{1/2}L_x^2$ since we may exchange derivatives in x for derivatives in v (and we gain in integrability around $t = 0$ when it is in this order). The variable v is defined on S^1, *i.e.* it is one dimensional so by Sobolev embedding we obtain the desired result.

Let us make a few comments. Proposition 1.1 shows that, at least in dimension two, it is possible to invert the order of the norms in x and v in (1.4) provided the exponent in x is not larger than two.
Since the space $L_v^{p_1}(L_x^{p_2})$ is included in the space $L_x^{p_2}(L_v^{p_1})$ for $p_1 \leq p_2$, inverting the order cannot lead to a better result than in Theorem 1.1. Moreover since $L_x^{p_2}(L_v^{p_1})$ is itself included in $L_{x,loc}^{p_1}(L_v^{p_1})$, the number of derivatives, which is gained in ρ, should be the same (provided f is at least as regular as g, as noted in the introduction). Hence the main question is under which condition we can have the same integrability for ρ.
But here it is easy to see that for the operator T_0^* to send L_x^p in $L_x^p(L_v^1)$, we need that $p \geq d$ (and the same for T_s^* of course). Indeed consider the function, for any $\eta \ll 1$

$$h(x) = \mathbb{I}_{|x|\leq\eta}.$$

Choosing the simple case $s = 0$ in the definition (4.1) of T_0^*, we have

$$T_0^*\, h(x,v) \sim e^{-|x|} \times \mathbb{I}_{|v\cdot x/|x||\leq\eta|x|}.$$

Therefore

$$\|h\|_{L^p} \sim \eta^{d/p}, \quad \|T_0^*\, h\|_{L_x^p(L_v^1)} \sim \eta,$$

and the requirement that $d/p \leq 1$. Since any estimate on T_0 implies by duality an averaging result, this corresponds to the condition $p \leq d^*$ for any equivalent of Proposition 1.1.
The estimate we derive in this subsection for T_s^* is a well known inequality about the so called Kakeya maximal function -see for example[12]. In here we prove it using Sobolev's embedding theorem as a consequence of the gain of 1/2 derivative. This does not work in higher dimensions where the problem is open.

4.2 Proofs of Theorems 1.2 and 1.3

We will have to bound as before

$$\int_{v\in S^1}\int_0^\infty g(x-vt,v)\frac{e^{-\lambda t}}{t^s}\,\phi(v)\,dt\,dv, \tag{4.2}$$

and

$$\int_{v\in S^1}\int_0^\infty f(x-vt,v)\frac{e^{-\lambda t}}{t^{s'}}\,\phi(v)\,dt\,dv. \tag{4.3}$$

We first note that thanks to the remark at the end of the proof of Theorem 1.1, we may take $s' = -1$ in (4.3). Then since $g(.,v)\phi(v)$ is even in v

$$\begin{aligned}\int_{v\in S^1}\int_0^\infty g(x-vt,v)\frac{e^{-\lambda t}}{t^s}\,\phi(v)\,dt\,dv &= \int_{v\in S^1}\int_0^\infty g(x+vt,v)\frac{e^{-\lambda t}}{t^s}\,\phi(v)\,dt\,dv\\ &= \frac{1}{2}\int_{v\in S^1}\int_{-\infty}^\infty g(x-vt,v)\frac{e^{-\lambda|t|}}{|t|^s}\,\phi(v)\,dt\,dv.\end{aligned}$$

Therefore for $s < -1/2$, we define T_s^* as in the previous subsection by (4.1), but for $s \geq -1/2$, we define

$$T_s^* h = \int_{-\infty}^{+\infty} h(x+vt)\,\frac{e^{-|t|}}{|t|^s}. \tag{4.4}$$

We use the notation $\tilde{T}_s^*$ for

$$\tilde{T}_s^* h = \int_0^{+\infty} h(x+vt)\,\frac{e^{-|t|}}{|t|^s}.$$

Theorem 1.2 is a direct consequence of the proposition

Proposition 4.1 *In dimension two, T_0^* and T_s^* with $s \leq -1/2$, are continuous from $L^p(\mathbb{R}^2)$ to $W_x^{\theta,p}(L_v^2)$ for any $\theta < 1/2$, provided $2 \leq p \leq 4$.*

This proposition implies the dual estimate for T_s, from $L_x^p(L_v^2)$ in $W_x^{\theta,p}$ with $\theta < 1/2$, $4/3 \leq p \leq 2$. For the proof of Theorem 1.2 from Prop. 4.1, we first interchange x and v derivatives as in Lemma 3.3, then we use the operator T_s^* and we conclude by a standard interpolation procedure as in Theorem 1.1. Therefore we omit this proof here and we give some details only for

Theorem 1.3 where the procedure is a bit more complicated. Proposition 4.1 is proved in subsection 4.3.

Theorem 1.3 is a consequence of the more precised proposition in dimension two

Proposition 4.2 *In dimension two, $\tilde{T}_0^*$ and T_s^*, for $s \leq -1$, are continuous from $L^4(\mathbb{R}^2)$ to $H_x^\theta(L_v^1)$ for any $\theta < 1/2$.*

It also requires the use of a proposition proved in [12] for the X-ray transform but which may easily be adapted here, namely

Proposition 4.3 *In dimension two, T_s^* with $s \leq 0$ is continuous from $L^2(\mathbb{R}^2)$ to $L_x^p(L_v^2)$ for any $2 \leq p < \infty$.*

This proposition for our operator is a trivial consequence from the one for the X-ray transform because it does not involve any derivative and our operator is pointwise bounded by the X-ray transform.
From these two propositions one may deduce by interpolation

Proposition 4.4 *In dimension two, for any $s \leq 0$, $1 < p_2 \leq 2$, $p_1 \geq p_2$, the operator T defined by (3.1) is continuous from $L_x^{p_2}(W_v^{s,p_1})$ to $\dot{W}_x^{s+\theta,r} + \dot{W}_x^{\theta,r}$ with norms $(C\lambda^{s+\theta-1-2/r+2/p_2}, C\lambda^{\theta-1-2/r+2/p_2})$ and*

$$\theta < \min\left(1/2,\, 2\left(1 - \frac{1}{p_2}\right)\right), \quad \frac{1}{r} = \frac{1}{2} + \frac{1}{2p_1} + \frac{2}{p_1}\left|\frac{3}{4} - \frac{1}{p_2}\right|.$$

This proposition is proved for $s \leq -1/2$ and $s = 0$, the general case being obtained by interpolation. The first step is to integrate by parts in v so as to be back to the operators T_s if $\lambda = 1$. Then for T_s we interpolate in p_1 between Prop. 4.1 and 4.2 if $p_2 = 4/3$. It is then enough to interpolate in p_2 with Prop. 4.3 first if $1 < p_2 < 4/3$ and with the known result in L^2 if $4/3 < p_2 < 2$. This proves Proposition 4.4 if $\lambda = 1$. To get the dependency on λ, we use a simple scaling argument

$$\begin{aligned}
\|Tf\|_{\dot W^{\delta,r}} &= \left(\int_x \left(\partial_x^\delta \int_v \int_0^\infty f(x-vt,v)\,\phi(\lambda t)\,dt\,dv\right)^r\,dx\right)^{1/r}\\
&= \lambda^{-1}\left(\int_x \left(\partial_x^\delta \int_v \int_0^\infty f(x-vu/\lambda,v)\,\phi(u)\,du\,dv\right)^r\,dx\right)^{1/r}\\
&= \lambda^{-1+\delta-d/r}\left(\int_y \left(\partial_y^\delta \int_v \int_0^\infty f((y-vu)/\lambda,v)\,\phi(u)\,dt\,dv\right)^r\,dy\right)^{1/r}\\
&\leq C\lambda^{-1+\delta-d/r}\left(\int_y \left(\int_v |\partial_v^s f(y/\lambda,v)|^{p_1}\,dv\right)^{p_2/p_1}\,dx\right)^{1/p_2}\\
&\leq C\lambda^{-1+\delta-d/r+d/p_2}\,\|f\|_{L_x^{p_2}(W_v^{s,p_1})}.
\end{aligned}$$

Now we apply Prop. 4.4 to λf and g solutions to (1.1). Note again that thanks to the arguments given at the end of the third section, we may have as much integrability in t as we want for f in the operator T and consequently the restriction $s \leq 0$ in Prop. 4.4 can be removed for f. As previously this gives us

$$\rho = \lambda\rho^1 + \rho^2,$$

with

$$\begin{aligned}
\|\rho^1\|_{\dot W_x^{\theta_f+\beta,r_f}} &\leq C\lambda^{\beta+\theta_f-1-d/r_f+d/p_2}\,\|f\|_{L_x^{p_2}(W_v^{\beta,p_1})},\\
\|\rho^2\|_{\dot W_x^{\theta_g+\gamma-\alpha,r_g}} &\leq C\lambda^{\gamma+\theta_g-1-d/r_g+d/q_2}\,\|g\|_{L_x^{q_2}(W_v^{\beta,q_1})},
\end{aligned}$$

where

$$\begin{aligned}
\theta_f &< \min\left(1/2, 2(1-1/p_2)\right), \quad \frac{1}{r_f} = \frac{1}{2} + \frac{1}{2p_1} + \frac{2}{p_1}\left|\frac{3}{4} - \frac{1}{p_2}\right|,\\
\theta_g &< \min\left(1/2, 2(1-1/q_2)\right), \quad \frac{1}{r_g} = \frac{1}{2} + \frac{1}{2q_1} + \frac{2}{q_1}\left|\frac{3}{4} - \frac{1}{q_2}\right|.
\end{aligned}$$

It only remains to do the interpolation in λ though the real method and that gives the formula of Theorem 1.3.

4.3 Proof of Proposition 4.1

We in fact show the following

Lemma 4.1 *For any set E and any $0 \le \theta < 1/2$, provided $s \le -1$ or $s = 0$*

$$\|\Delta_x^{\theta/2} T_s^* \mathbb{I}_E\|^4_{L^4_x(\mathbb{R}^2,\, L^2_v(S^1))} \le C\,|E|. \tag{4.5}$$

This implies the corresponding estimates with norms of Lorentz spaces for any function and by Sobolev embedding ($\theta < 1/2$) the proposition. For an example of how to pass from Lemma 4.1 to Prop. 4.1 we refer the reader to the end of the first appendix, where the procedure is used for the "classical" L^2 estimate.

Proof of Lemma 4.1. First of all, we decompose the sphere S^1 into subdomains S_k with $k = 1,\, 2$ such that $|v_k| > 1/2$ in S_k. Of course it is enough to prove (4.5) with S_k instead of S^1 and by symmetry we do it only for S_1. Now we are going to make two reductions.

Step 1: Reduction to the compactly supported case.

We explain why it is enough to prove for any $K > 0$ and any set $E \in B(0, K)$, the inequality

$$\|\Delta_x^{\theta/2} T_s^* \mathbb{I}_E\|^4_{L^4_x(B(0,K),\, L^2_v(S_1))} \le C(K)\,|E|. \tag{4.6}$$

Take any set $E \subset \mathbb{R}^2$ with finite measure and any $K > 0$. We decompose E into $\cup_i E_i$ with $E_i \subset B(x_i, K)$ and $|x_i - x_j| > K/2$ and $E_i \cap E_j = \emptyset$, $\forall i \ne j$. Then

$$\mathbb{I}_E(y) = \sum_i \mathbb{I}_{E_i}(y),$$

and consequently

$$T_s^* \mathbb{I}_E(x, v) = \sum_i T_s^* \mathbb{I}_{E_i}(x, v)\ \mathbb{I}_{B(x_i, 2K)}(x) + \sum_i T_s^* \mathbb{I}_E(x, v)\ \mathbb{I}_{|x - x_i| > 2K} = I + II.$$

Now, of course because of the condition $|x_i - x_j| > K/2$

$$\int_{\mathbb{R}^2} \left(\int_{S_1} |\Delta_x^{\theta/2}\, I|^2\, dv \right)^2 dx = C \sum_i \int_{B(x_i, 2K)} \left(\int_{S_1} |\Delta_x^{\theta/2}\, T_s^* \mathbb{I}_{E_i}(x, v)|^2\, dv \right)^2 dx$$
$$\le C(2K) \sum_i |E_i| \le C(2K)\,|E|,$$

since (4.6) is obviously invariant by translation and hence true as well if we replace $B(0, K)$ by $B(y, K)$ for any y.

As for the second term, we remark that, as $E_i \subset B(x_i, K)$

$$T_s^* \mathbb{I}_{E_i}(x,v)\; \mathbb{I}_{|x-x_i|>2K} \leq e^{-|x-x_i|/2-K/2},$$

and that furthermore (that inequality is proved in [12]), for any x

$$\int_{S_1} |T_s^* \mathbb{I}_{E_i}(x,v)|^2\, dv \leq C\, |E_i|.$$

Eventually we simply bound in L^4

$$\begin{aligned}\int_{\mathbb{R}^2} \left(\int_{S_1} |II|^2\, dv \right)^2 dx &\leq C\, e^{-K} \sum_{i,j} |E_i|^{1/2}\, |E_j|^{1/2} \int_{\mathbb{R}^2} e^{-|x-x_i|/2-|x-x_j|/2}\, dx \\ &\leq C\, e^{-K}\, |E|.\end{aligned}$$

We have decomposed $T_s^* \mathbb{I}_E$ into two terms for any K. The first one belongs to $W_x^{\theta,4}(L_v^2)$ with norm $(C(2K)\,|E|)^{1/4}$ (which is obviously at most polynomial in K) and the second one in L^4 with norm $e^{-K/4}\,|E|^{1/4}$. By real interpolation, we deduce that $T_0^* \mathbb{I}_E$ belongs to $W_x^{\theta',4}(L_v^2)$ with norm $C|E|^{1/4}$ for any $\theta' < \theta$, which is exactly what we want.

Step 2: Reduction to the X-ray transform.
The aim here is to get back the case where $T_s^* \mathbb{I}_E(x,v)$ is invariant along any line with direction v like the X-ray transform. So first of all, we write

$$\begin{aligned}|\Delta_x^{\theta/2} T_s^* \mathbb{I}_E(x,v)| &= |\Delta_x^{\theta/2} \int_{-\infty}^{0} v \cdot \nabla_x T_s^* \mathbb{I}_E(x+tv,v)\, dt| \\ &\leq \int_{-\infty}^{+\infty} |\Delta_x^{\theta/2}\, v \cdot \nabla_x T_s^* \mathbb{I}_E(x+tv,v)|\, dt.\end{aligned}$$

All these expressions make sense because now $E \subset B(0,K)$ and because $v \cdot \nabla_x T_s^* \mathbb{I}_E(x+tv,v)$ is if $s < -1/2$

$$\begin{aligned}v \cdot \nabla_x T_s^* \mathbb{I}_E(x+tv,v) &= \int_0^\infty v \cdot \nabla_x \mathbb{I}_E(x+tv+rv) \frac{e^{-r}}{r^s}\, dr \\ &= \int_0^\infty \frac{\partial}{\partial r} \big(\mathbb{I}_E(x+tv+rv) \big) \frac{e^{-r}}{r^s}\, dr \qquad (4.7) \\ &= \int_0^\infty \mathbb{I}_E(x+tv+rv)\; \left(\frac{s\, e^{-r}}{r^{s+1}} - \frac{e^{-r}}{r^s} \right)\, dr,\end{aligned}$$

by integration by parts in r and because $s < -1/2$ and as a consequence r^{-s} vanishes at $r = 0$. If $s = -1/2$ or $s = 0$, then T_s^* is the integral on the whole line by (4.4) and so

$$v \cdot \nabla_x T^*_{-1/2} \mathbb{I}_E(x+tv, v) = \frac{1}{2} T^*_{1/2} \mathbb{I}_E(x+tv) + \int_{-\infty}^{\infty} \mathbb{I}_E(x+tv+rv) \frac{e^{-|r|}}{|r|^{1/2}} \times \frac{r}{|r|}\, dr,$$
$$v \cdot \nabla_x T^*_0 \mathbb{I}_E(x+tv, v) = \int_{-\infty}^{\infty} \mathbb{I}_E(x+tv+rv)\, e^{-|r|} \times \frac{r}{|r|}\, dr. \tag{4.8}$$

Now we denote

$$T\mathbb{I}_E(x, v) = \int_{-\infty}^{+\infty} |\Delta_x^{\theta/2}\, v \cdot \nabla_x T_s^* \mathbb{I}_E(x+tv, v)|\, dt.$$

Thanks to (4.7) and (4.8), we know the following properties on T, for some $\theta' > 0$ (in fact $\theta' = 1/2 - \theta$)

$$v \cdot \nabla_x T\, \mathbb{I}_E(x, v) = 0, \quad \|\Delta_x^{\theta'/2}\, T\, \mathbb{I}_E\|_{L^2_{B(0,K)\times S_1}} \le C\, |E|^{1/2}. \tag{4.9}$$

Note that here we need the condition $s \le -1/2$ or $s = 0$ because the gain of half a derivative for T_s^* is possible only if $s \le 1/2$ and from (4.7) and (4.8), we see that we work in fact with $s+1$ if $s \neq 0$.
We want to deduce from (4.9)

$$\|T\mathbb{I}_E\|^4_{L^4_x(B(0,K),\, L^2_v(S_1))} \le C(K)\, |E|. \tag{4.10}$$

Step 3: Deduction of (4.10) *from* (4.9).
We begin with

$$\begin{aligned}
\|T\mathbb{I}_E\|^4{}_{L^4_x(B(0,K),\, L^2_v(S_1))} &= \int_{B(0,K)} \left(\int_{v\in S_1} |T\mathbb{I}_E(x, v)|^2\, dv \right)^2 dx \\
&= \int_{B(0,K)} \int_{v,w\in S_1} |T\mathbb{I}_E(x, v)|^2 \times |T\mathbb{I}_E(x, w)|^2\, dv\, dw\, dx \\
&= \int_{v\in S_1} \int_{x\in B(0,K)} \int_{w\in S_1} |T\mathbb{I}_E(x, v)|^2\, |T\mathbb{I}_E(x, w)|^2\, dw\, dx\, dv.
\end{aligned}$$

We change variables in x decomposing x in $y + lv$ with y in the plane H_1 of equation $x_1 = 0$. Since $|v_1| > 1/2$, the jacobian of the transformation is bounded and as all the terms in the integral are non negative, we may simply bound

$$\|T\mathbb{I}_E\|^4_{L^4_x(B(0,K),\,L^2_v(S^1))} \leq \int_{v\in S_1}\int_{y\in H_1}\int_{l=-K}^{K}\int_{w\in S_1} |T\,\mathbb{I}_E(y+lv,v)|^2 \times |T\mathbb{I}_E(y+lv,w)|^2\,dw\,dl\,dy\,dv$$

$$\leq \int_{v\in S_1}\int_{y\in H_1} |T\,\mathbb{I}_E(y,v)|^2 \times \left(\int_{l=-K}^{K}\int_{w\in S_1} |T\,\mathbb{I}_E(y+lv,w)|^2 dw\,dl\right)\,dy\,dv,$$

because $Tf(x,v)$ is constant on any line with direction v and therefore $T\,\mathbb{I}_E(y+lv,v)$ does not depend on l. We denote

$$I(y,v) = \int_{l=-K}^{K}\int_{w\in S_1} |T\mathbb{I}_E(y+lv,w)|^2\,dw\,dl,$$

and we want to show that I belongs to L^∞. So we fix y and v and we first decompose S_1 into the union of S_1^i with $S_1^i = \{w\in S^1,\ 2^{-i-1} < |v-w| < 2^{-i}\}$ and so

$$I(l,v) = \sum_{i=0}^{\infty} I_i(l,v) = \sum_{i=0}^{\infty}\int_{l=-K}^{K}\int_{w\in S_1^i} |T\,\mathbb{I}_E(y+lv,w)|^2\,dw\,dl.$$

Of course $T\,\mathbb{I}_E(y+lv,w)$ is constant along any line with direction w so we may bound

$$I_i \leq \frac{1}{2K}\int_{w\in S_1^i}\int_{l=-K}^{K}\int_{s=-K}^{K} |T\,\mathbb{I}_E(y+sw+lv,w)|^2\,ds\,dl\,dw.$$

We change again variables from l and s to $z = y+sw+lv$. We denote by $C_{y,v,w}$ the set $\{y+sw+lv,\ |s|\leq K,\ |l|\leq K\}$ and by $|(v,w)|$ the sinus of the angle between v and w. Then

$$I_i \leq \frac{1}{2K}\int_{w\in S_1^i}\int_{z\in C_{y,v,w}} |T\,\mathbb{I}_E(z,w)|^2 \frac{dz\,dw}{|(v,w)|}$$

$$\leq \frac{2^{i+1}}{2K}\int_{w\in S_1^i}\int_{z\in C_{y,v,w}} |T\,\mathbb{I}_E(z,w)|^2\,dz\,dw.$$

Denote $C_{y,v} = \bigcup_{w\in S_1^i} C_{y,v,w}$ and $\tilde{E} = E\cap C_{y,v}$. Clearly, as all the terms are non negative

$$I_i \leq \frac{2^{i+1}}{2K}\int_{w\in S_1^i}\int_{z\in C_{y,v}} |T\,\mathbb{I}_{\tilde{E}}(z,w)|^2\,dz\,dw.$$

Using a Hölder estimate, we find for any $p > 2$,

$$I_i \le \frac{2^{i+1}}{2K} \times |C_{y,v}|^{1-2/p} \times \int_{w\in S_1^i}\left(\int_{z\in C_{y,v}} |T\,\mathbb{I}_{\tilde{E}}(z,w)|^p\,dz\right)^{2/p} dw$$
$$\le C(K)\,2^{i+1} \times 2^{-i(1-2/p)} \times \int_{w\in S^1}\left(\int_{z\in B(0,2K)} |T\,\mathbb{I}_{\tilde{E}}(z,w)|^p\,dz\right)^{2/p} dw,$$

because the measure of $C_{y,v}$ is bounded by a constant depending on K times 2^{-i}. Now by Sobolev embedding, for $1/2-\theta'/2 \le 1/p < 1/2$, the last integral is dominated by the $L^2_w H^{\theta'}_z$ norm of $T\mathbb{I}_{\tilde{E}}$. Therefore, taking $1/p = 1/2-\theta'/2$, we get by (4.9)

$$I_i \le C(K)\,2^{i+1} \times 2^{-i\theta'} \times \int_{w\in S^1}\int_{z\in B(0,2K)} |\Delta_x^{\theta'/2} T\,\mathbb{I}_{\tilde{E}}(z,w)|^2\,dz\,dw$$
$$\le C(K)\,2^{i+1} \times 2^{-i\theta'} \times C\,|\tilde{E}| \le C(K) \times 2^{-i\theta'},$$

because the measure of $\tilde{E}$ is less than the measure of $C_{y,v}$. Eventually we may sum up the series and get

$$I = \sum_{i=0}^{\infty} I_i \le C(K).$$

This has as immediate consequence that

$$\|\Delta_x^{s/2} T\mathbb{I}_E\|^4_{L^4_x(B(0,K),\,L^2_v(S_1))} \le C(K)\int_{v\in S_1}\int_{y\in H_1} |\Delta_x^{s/2} T\,\mathbb{I}_E(y,v)|^2\,dy\,dw$$
$$\le C(K) \times |E|,$$

using again the known L^2 estimate (4.9) on T. □

Note that it is relatively simple to find a set E for which the lemma would be false if $p > 4$ in dimension two. Indeed, one may take for example a set composed of the N sets E_i of equations in polar coordinates r, θ, $\theta \in [i/N,\ i/N + i/2N]$ and $r \le 1$. Then $|E| \ge 1$ and for any x in the square of size $1/N$ centered at the origin $\int_v |\Delta_x^{1/4}\mathbb{I}_E(x,v)|^2\,dv = N$ and so to have

$$N^{-2} \times N^p \le \int_{B(0,2K)} \left(\int_v |\Delta_x^{1/4}\,\mathbb{I}_E(x,v)|^2\,dv\right)^{p/2} dx \le CN^{p/2},$$

one must have $p \le 4$. So in this sense Proposition 4.1 is optimal.

4.4 Proof of Proposition 4.2

Let us first remark that Proposition 4.2 can be proved with the same method as for Proposition 4.1. Indeed it is enough to bound

$$\int_{B(0,K)} \left(\int_{v\in S_1} |T\mathbb{I}_E(x,v)|\, dv \right)^4 dx \le \int_{B(0,K)} \int_{v\in S_1} |T\mathbb{I}_E(x,v)|^2 \times \left(\int_{w\in S_1} |T\mathbb{I}_E(x,w)|\, dw \right)^2 dv\, dx,$$

and then for any $k<1$

$$\int_{B(0,K)} \left(\int_{v\in S_1} |T\mathbb{I}_E(x,v)|\, dv \right)^4 dx \le \int_{B(0,K)} \int_{v\in S_1} |T\mathbb{I}_E(x,v)|^2 \times \left(\int_{w\in S_1} |T\mathbb{I}_E(x,w)|^2 \times |(v,w)|^k\, dw \right) \times \left(\int_w |(v,w)|^{-k} dw \right) dv\, dx.$$

That gives almost an additional $|(v,w)|$ which is just what is needed to go from Prop. 4.1 to Prop. 4.2.
We note as well that the same counterexample as in the previous subsection holds here.

However the previous method makes necessary the evenness condition on $g\phi$ and so we present another proof, using a TT^* argument, which does not require it. We denote by the general notation T all the operators for $s\le -1$ or $s=0$

$$Tf(x,v) = \int_0^\infty f(x+vt)\, \frac{e^{-t}}{t^s}\, dt.$$

Proposition 4.2 is equivalent by duality to

$$\|Tf\|_{W_x^{s,4}(\mathbb{R}^2,\ L_v^1(S^1))} \le C\, \|f\|_{L^2}, \quad \forall s<1/2. \tag{4.11}$$

Step 1: Reduction to the compactly supported case. The procedure is the same as in the previous case so we omit it. It enables to deduce (4.11) from the inequality, for any f compactly supported in $B(0,1)$

$$\|Tf\|_{W_x^{s,4}(B(0,1),\ L_v^1(S^1))} \le C\, \|f\|_{L^2}, \quad \forall s<1/2.$$

Therefore we may define for some function $\psi(t) \in C^1_c$ with $\psi(t) = t^{-s}\, e^{-t}$ if $0 \leq t \leq 1$ and ψ with compact support in $[0,\ 2]$

$$\tilde{T} f(x,v) = \int_0^\infty f(x+vt)\,\psi(t)\,dt,$$

and it is enough to show that for any f compactly supported in $B(0,1)$

$$\|\tilde{T} f\|_{W^{s,4}_x(B(0,1),\ L^1_v(S^1))} \leq C\,\|f\|_{L^{4/3}_x(L^\infty_v)}, \quad \forall s < 1/2. \tag{4.12}$$

Step 2: The $T\,T^*$ argument. The last inequality is equivalent to show that for any function f of the two variables x, v

$$\|\tilde{T}\,\tilde{T}^*\,f\|_{W^{1,4}_x(\mathbb{R}^2,\ L^1_v(S^1))} \leq C\,\|f\|_{L^{4/3}_x(L^\infty_v)}.$$

Then we perform a cut-off in frequency space. Take $K \in \mathcal{S}(\mathbb{R})$ with $\hat{K}$ supported in $[-1,1]$, $N > 1$ and define for any function f the $f_N(x) = N^2\,K(N|x|) \star f$. The last estimate is implied by

$$\|\tilde{T}\,\tilde{T}^* f_N\|_{W^{1,4}_x(\mathbb{R}^2,\ L^1_v(S^1))} \leq C\,\ln N\,\|f\|_{L^{4/3}_x(L^\infty_v)}. \tag{4.13}$$

We note that

$$\begin{aligned}\tilde{T}\,\tilde{T}^* f_N(x,v) &= \int_{w\in S^1}\int_0^\infty\int_0^\infty f_N(x+vt-ws,w)\,\psi(s)\,\psi(t)\,ds\,dt\,dw\\ &= \int_{w\in S^1} S\,f_N(x,v,w)\,dw.\end{aligned}$$

We also perform a dyadic decomposition of S^1, introducing again the $S^i_1 = \{w \in S^1 \mid 2^{-i-1} < |(v,w)| \leq 2^{-i}\}$ for $i < \ln N$ and $S_0 = \{w \in S^1 \mid |(v,w)| \leq 1/N\}$. Consequently

$$\begin{aligned}\tilde{T}\,\tilde{T}^* f_N(x,v) = R_0\,f_N + \sum_{i=1}^{\ln N} R_i\,f_N &= \int_{w\in S_0} S\,f_N(x,v,w)\,dw\\ &\quad+\sum_{i=1}^{\ln N}\int_{w\in S^i_1} S\,f_N(x,v,w)\,dw.\end{aligned}$$

Moreover by integration by parts in t and s

$$\begin{aligned} v\cdot\nabla_x S\, f(x,v,w) &= -\int_0^\infty f(x-ws,w)\,\psi(s)\,ds \\ &\quad -\int_0^\infty\int_0^\infty f(x+vt-ws,w)\,\psi(s)\,\psi'(t)\,ds\,dt, \\ w\cdot\nabla_x S\, f(x,v,w) &= \int_0^\infty f(x+vt,w)\,\psi(t)\,dt \\ &\quad +\int_0^\infty\int_0^\infty f(x+vt-ws,w)\,\psi'(s)\,\psi(t)\,ds\,dt. \end{aligned}$$

Since (we recall that (v,w) is the sinus of the angle between v and w)

$$|\nabla_x\, S\, f(x,v,w)| \le \frac{C}{|(v,w)|}\,\big(|v\cdot\nabla_x S\, f| + |w\cdot\nabla_x f|\big),$$

we may bound

$$|\nabla_x R_i\, f_N(x,v)| \le |\tilde T\,\sup_w f_N| + 2^{i+1}\,|\tilde T_i^*\, f_N| + 2^{i+1}\int_{w\in S_1^i} |\tilde S\, f_N(x,v,w)|\,dw,$$

where

$$\begin{aligned} &\tilde S\, f_N(x,v,w) = \int_0^\infty\int_0^\infty f_N(x+vt-ws,w)\Phi(s,t)\,ds\,dt, \\ &\Phi = |\psi'(s)|\,\psi(t) + \psi(s)\,|\psi'(t)|, \\ &\tilde T_i^*\, f_N(x,v) = \int_{w\in S_1^i}\int_0^\infty f_N(x-ws,w)\,ds\,dw. \end{aligned}$$

Step 3: Bound on the terms coming from $\tilde T$ and $\tilde T_i^*$. Denote X the X-ray transform

$$X\, h(x,v) = \int_{-\infty}^\infty h(x+vt)\,dt.$$

We start with $\tilde T$

$$\|\tilde T\,\sup_w f_N\|_{L^4_x(L^1_v)} \le \|X\,\sup_w f_N\|_{L^4_x(L^1_v)} \le C\|\sup_w f_N\|_{L^{4/3}} \le C\|f_N\|_{L^{4/3}_x(L^\infty_v)},$$

where we have used the bound for the X-ray transform proved in [12]. Now for T_i^*

$$
\begin{aligned}
\|T_i^* f_N\|_{L^4_x(L^1_v)} &\le \left\| \int_{v\in S^1}\int_{w\in S^i_1}\int_0^\infty |f_N(x-ws,w)|\,ds\,dw\,dv \right\|_{L^4_x} \\
&\le 2^{-i} \left\| \int_{w\in S^1}\int_0^\infty |f_N(x-ws,w)|\,ds\,dw \right\|_{L^4_x} \\
&\le 2^{-i}\|X^*\,|f_N|\|_{L^4_x} \le C\,2^{-i}\,\|f_N\|_{L^{4/3}_x(L^\infty_v)},
\end{aligned}
$$

using again the estimate for X in [12].

Step 4: Bound on the term from $\tilde S$. We first estimate

$$
|\tilde S\, f_N(x,v,w)| \le \int_0^\infty\int_0^\infty \sup_z f_N(x-vt+ws,z)\,dt\,ds,
$$

then we change variable denoting $r = v^\perp\cdot(ws-vt)$ and

$$
\begin{aligned}
\mathbb{I}_{w\in S^i_1}|\tilde S\, f_N(x,v,w)| &\le \int_{|r|\le 2^{-i+2}}\int_{-\infty}^\infty \sup_z f_N(x+rv^\perp+ws,z)\,ds\,dr \\
&\le 2^i\int_{y\in B(0,2^{-i+2})} X(\sup_z|f_N(.,z)|)(x+y,w)\,dy.
\end{aligned}
$$

Finally changing the order of integration

$$
\begin{aligned}
&\left\| \int_{v\in S^1}\int_{w\in S^i_1} |\tilde S\, f_N(x,v,w)|\,dw\,dv \right\|_{L^4_x} \\
&\quad\le 2^i\int_{y\in B(0,2^{-i+2})} \left\| \int_{w\in S^1}\int_{v\in S^1_i} X(\sup_z|f_N(.,z)|)(x+y,w)\,dv\,dw \right\|_{L^4_x} dy \\
&\quad\le \int_{y\in B(0,2^{-i+2})} \|X(\sup_z|f_N(.,z)|)(x+y,v)\|_{L^4_x(L^1_v)}\,dy,
\end{aligned}
$$

and we obtain

$$
\begin{aligned}
\left\| \int_{v\in S^1}\int_{w\in S^i_1} |\tilde S\, f_N(x,v,w)|\,dw\,dv \right\|_{L^4_x} &\le \int_{y\in B(0,2^{-i+2})} \|\sup_z f_N(x+y,z)\|_{L^{4/3}}\,dy \\
&\le C\,2^{-i}\,\|f_N\|_{L^{4/3}_x(L^\infty_v)}.
\end{aligned}
$$

Step 5: Bound on R_0. We simply differentiate under the integral

$$\|\nabla_x R_0\, f_N\|_{L^4_x(L^1_v)} \leq N \left\| \int_{S_0}\int_0^\infty\int_0^\infty |f(x+vt-ws,w)|\,\psi(s)\,\psi(t)\,ds\,dt\,dw \right\|_{L^4_x(L^1_v)}.$$

For this last term the same proof as for $\tilde{S}$ shows that

$$\|\nabla_x R_0\, f_N\|_{L^4_x(L^1_v)} \leq \|f\|_{L^{4/3}_x(L^\infty_v)}.$$

Having proved that every $R_i\, f_N$ are bounded in $W^{1,4}_x(L^1_v)$ by $\|f\|_{L^{4/3}_x(L^\infty_v)}$, we deduce (4.13), which concludes the proof.

Appendix A: A direct proof for Proposition 3.1

We present here a direct method in L^2 for the dual operator T^*. More precisely, we show

Proposition 4.5 *Let T^*_s be defined by* (4.1). *Then this operator is continuous from L^p_x in $L^p_v W^{\theta,p}_x$ for $\theta < 1-1/\bar{p}$ with $\bar{p} = \min(p,p^*)$ provided $s < 1/2$.*

We do not indicate here how one may deduce from that Prop. 3.1 in the case $p_1 = p_2$. The procedure is fairly obvious, it is enough to exchange first derivatives in v for derivatives in x (thus losing integrability in t hence the need for T_s and not only T_0) then apply Prop. 4.5. Note that the assumption $s < 1/2$ implies that $s+\theta < 1$.

In the spirit of [12], we first prove Proposition 4.5 for characteristic functions of sets. Since the proof is more complex, it is convenient to treat first only the case of simple sets. The first point to note is that we may work in a domain S_0 in v which is included in $\{v \in S^{d-1},\ 1/4d < v_i < 1/2\ \forall i \leq d\}$ instead of working in the whole sphere since the sphere may be decomposed in a finite number of domains of the same form as S_0 and the result is the same on any of them due to the invariance by rotation of the problem.
Thus for any $N > 0$, we say that a set E belongs to $\mathcal{C}_N$ if it is the union of closed squares (or cubes or hypercubes) of the form $[i_1/N,\ i_1/N + 1/N] \times$

$\ldots \times [i_d/N,\ i_d/N+1/N]$ where $i_1,\ldots,i_d$ are integers. Of course we choose this form for $\mathcal{C}_N$ because the "bad" directions which are along the axis of coordinates do not belong to S_0. Then we prove

Lemma 4.2 *For any $N>0$ and any $E\in\mathcal{C}_N$, we have for $\theta<1-1/\bar p$ with $\bar p=\min(p,p^*)$ and $s<1/2$*

$$\|T_s^*\mathbb{I}_E\|^p_{L^p_v(S_0,\ W^{\theta,p}_x)}\leq C|E|.$$

Proof. We compute directly the norm using the formula

$$\|T_s^*\mathbb{I}_E\|^p_{L^p_vW^{\theta,p}_x}=\int_{x,\,y\in\mathbb{R}^d}\int_{v\in S_0}|T_s^*\mathbb{I}_E(x,v)-T_s^*\mathbb{I}_E(y,v)|^p|x-y|^{-d-\theta p}\,dv\,dy\,dx.$$

Let us decompose according to the distance between x and y

$$\begin{aligned}\|T_s^*\mathbb{I}_E\|^p_{L^p_vW^{\theta,p}_x}=&\int_{|x-y|\geq 1}\int_{v\in S_0}|T_s^*\mathbb{I}_E(x,v)-T_s^*\mathbb{I}_E(y,v)|^p|x-y|^{-d-\theta p}\,dv\,dy\,dx\\&+\sum_{i=1}^{\infty}\int_{2^{-i}\leq|x-y|<2^{-i+1}}\ldots\end{aligned}$$

Of course the first term is dominated by the power p of the norm of $T_s^*\mathbb{I}_E$ in $L^p_{x,v}$ which is trivially bounded by the measure of E (see the proof of Theorem 1.1 for instance). Since we do not want to get the precised critical case $\theta=1-1/\bar p$, it is therefore enough to show that for any M

$$\int_{1/M\leq|x-y|<2/M}\int_{v\in S_0}|T_s^*\mathbb{I}_E(x,v)-T_s^*\mathbb{I}_E(y,v)|^pM^{d+\theta p}\,dv\,dy\,dx\leq C|E|. \tag{4.14}$$

The first point to note, is that we may limit ourselves to the case where E has a fixed bounded diameter K independent on M or i and where we integrate over a ball of the same diameter. Indeed let us fix a ball, then

$$\begin{aligned}&\int_{x\in B(x_0,K)}\int_{1/M\leq|x-y|<2/M}\int_{v\in S_0}|T_s^*\mathbb{I}_E(x,v)-T_s^*\mathbb{I}_E(y,v)|^pM^{d+\theta p}\,dv\,dy\,dx\\&\leq C\int_{B(x_0,K)}\int_{1/M\leq|x-y|<2/M}\int_v|T_s^*\mathbb{I}_{E\cap B(x_0,2K)}(x,v)-T_s^*\mathbb{I}_{E\cap B(x_0,2K)}(y,v)|^pM^{d+\theta p}\\&\quad+Ce^{-K}\int_{B(x_0,K)}\int_{1/M\leq|x-y|<2/M}\int_v(|T_s^*\mathbb{I}_E(x,v)|^p+|T_s^*\mathbb{I}_E(y,v)|^p)M^{d+\theta p},\end{aligned}$$

because of the e^{-t} term in T_s^* of course. If we are able to prove that for $\theta' > \theta$ but with $\theta' < 1 - 1/\bar{p}$

$$\int_{B(x_0,K)} \int_{1/M \le |y-x| < 2/M} \int_v |T_s^* \mathbb{I}_{E\cap B(x_0,2K)}(x,v) - T_s^* \mathbb{I}_{E\cap B(x_0,2K)}(y,v)|^p M^{d+\theta' p} \le C_K |E \cap B(x_0, 2K)|, \tag{4.15}$$

summing on the balls, we get

$$\begin{aligned}
&\int_{x\in\mathbb{R}^d} \int_{1/M\le|x-y|<2/M} \int_{v\in S_0} |T_s^*\mathbb{I}_E(x,v) - T_s^*\mathbb{I}_E(y,v)|^p M^{d+\theta p}\, dv\, dy\, dx \\
&\le C_K M^{\theta-\theta'} |E| \\
&\quad + Ce^{-K} \int_{\mathbb{R}^d} \int_{1/M\le|x-y|<2/M} \int_v (|T_s^*\mathbb{I}_E(x,v)|^p + |T_s^*\mathbb{I}_E(y,v)|^p) M^{d+\theta p} \\
&\le C_K M^{\theta-\theta'} |E| + Ce^{-K} M^{d+\theta p} |E|.
\end{aligned}$$

A simple scaling argument shows that, in (4.15), C_K is dominated by a power of K (depending on p). So choosing eventually K in terms of M we may deduce (4.14) from (4.15). Hence from now on, E will have a given finite diameter and the integrals in x or y will be taken inside a ball.

Before proving (4.15), we remark that we may choose $M = N$ (not a great surprise). If $E \in \mathcal{C}_N$ then E belongs to every $\mathcal{C}_{2^i N}$ simply by dividing each hypercube in 2^{di} smaller identical hypercubes: So we may always take $N \ge M$. And if (4.15) is true for $M = N$, it is true for all $M \le N$ since for instance

$$\begin{aligned}
&\int_{2/N\le|x-y|<4/N} \int_v |T_s^*\mathbb{I}_E(x,v) - T_s^*\mathbb{I}_E(y,v)|^p \left(\frac{N}{2}\right)^{d+\theta p} dv\, dy\, dx \\
&\le C_p \int_{2/N\le|x-y|<4/N} \int_v |T_s^*\mathbb{I}_E(x,v) - T_s^*\mathbb{I}E(x+(y-x)/2,v)|^p \left(\frac{N}{2}\right)^{d+\theta p} \\
&\quad + C_p \int_{2/N\le|x-y|<4/N} \int_v |T_s^*\mathbb{I}_E(x+(y-x)/2,v) - T_s^*\mathbb{I}E(y,v)|^p \left(\frac{N}{2}\right)^{d+\theta p} \\
&\le \frac{2\, C_p}{2^{d+\theta p}} N^{\theta p-\theta' p} \int_{1/N\le|x-y|<2/N} \int_v |T_s^*\mathbb{I}_E(x,v) - T_s^*\mathbb{I}_E(y,v)|^p N^{d+\theta' p}\, dv\, dy\, dx,
\end{aligned}$$

where C_p is such that $|a+b|^p \le C_p\,|a|^p + C_p\,|b|^p$. Then $2\,C_p N^{\theta p-\theta' p}$ is less than 1 (unless N is of order one but the proof is trivial then) if $\theta' \ge \theta + C/\ln N$.

So (4.15) for $M = N$ implies (4.15) for $M = N/2$ and by repeating the same argument $\ln N/\ln\ln N$ times, for $\ln N \le M \le N$ with a final number of derivatives equal to $\theta_f = \theta_0 - C/\ln\ln N$, which is all right. Now of course if $M \le \ln N$ then the argument is obvious because we may lose at most a $\ln N$ factor which does not matter.

The last reduction of the problem we make is to regularize T_s^*. Indeed by the same kind of argument, we may take T_s^* of the form

$$T_s^* \mathbb{I}_E = \int_0^\infty \mathbb{I}_E(x+vt) \frac{e^{-t}}{(1/N+t)^s}\, dt,$$

and denoting C_i, $1 \le i \le n$, the hypercubes which compose E and x_i their center, we approximate $T_s^* \mathbb{I}_E$ by

$$\begin{aligned} T_N(x,v) &= \sum_{i=1}^n l_i(x,v)\phi_i(x), \\ l_i(x,v) &= \int_0^\infty \mathbb{I}_{C_i}(x+vt)\, dt, \quad \phi_i(x) = \frac{e^{-|x-x_i|}}{(1/N+|x-x_i|)^s}. \end{aligned}$$

We may do so because

$$|T_N(x,v) - T_s^* \mathbb{I}_E(x,v)| \le C \int_0^\infty \mathbb{I}_E(x+vt) N^{s-1} \frac{e^{-t}}{1/N+t}\, dt.$$

Therefore since $s + \theta < 1$, we have

$$\int_{2/N \le |x-y| < 4/N} \int_v |(T_s^* \mathbb{I}_E - T_N)(x,v)|^p N^{d+\theta p}\, dv\, dy\, dx \le C \|T_1^* \mathbb{I}_E\|^p_{L^p_{x,v}} \le C|E|,$$

and in proving (4.15), we may replace $T_s^* \mathbb{I}_E$ by T_N.

Estimate (4.15) is a consequence of the following

$$\begin{aligned} \sup_{|\xi| \le 1} \int_{B(0,K)} \int_{v \in S_0} |\nabla_x T_N(x+\xi,v)|^p\, dv\, dx \\ \le \int_{B(0,2K)} \int_{v \in S_0} |\nabla_x T_N(x,v)|^p\, dv\, dx \le N^{p-\theta p}\, |E|. \end{aligned} \tag{4.16}$$

Indeed, writing

$$\begin{aligned}|T_N(x,v)-T_N(y,v)| &= \left|\int_0^1 (y-x)\nabla_x T_N(x+s(y-x),v)\,ds\right|\\ &\le |x-y|\times\int_0^1 |\nabla_x T_N(x+s(y-x),v)|\,ds,\end{aligned}$$

and inserting this in the left hand side of (4.15), we find after a simple Hölder estimate in s

$$\begin{aligned}&\int_{B(0,K)}\int_{1/N\le|y-x|<2/N}\int_v |T_s^*\mathbb{I}_{E\cap B(x_0,2K)}(x,v)-T_s^*\mathbb{I}_{E\cap B(x_0,2K)}(y,v)|^p N^{d+\theta p}\\ &\le \int_0^1\int_{B(x_0,K)}\int_{1/N\le|\xi|<2/N}\int_v |\nabla_x T_N(x+s\xi,v)|^p N^{d+\theta p-p}\,dv\,dy\,dx\,ds\\ &\le \int_0^1\int_{|\xi|\le 2/N}\int_{B(x_0,K)}\int_v |\nabla_x T_N(x+s\xi,v)|^p N^{d+\theta p-p}\,dv\,dx\,dy\,ds \le C|E|,\end{aligned}$$

if (4.16) holds. To prove (4.16), we compute the derivative of T_N which may be decomposed into

$$\begin{aligned}|\nabla_x T_N(x,v)| = \left|\sum_{i=1}^n \nabla_x l_i(x,v)\phi_i(x)+l_i(x,v)\nabla_x\phi_i(x)\right| &\le \left|\sum_i \nabla_x l_i(x,v)\phi_i(x)\right|\\ &+ CN^s\sum_i l_i(x)\frac{e^{-|x-x_i|}}{1/N+|x-x_i|}.\end{aligned}$$

The last term poses no problem, it leads to the same computation as for the approximation of $T_s^*\mathbb{I}_E$ by T_N (as $s+\theta<1$) and so we do not repeat it here. We focus on the first term instead.

It is easy to compute $\nabla_x l_i$. It has a non zero component only in the space orthogonal to v. We denote by $L(x,v)$ the line passing through x and of direction v and by $n_i^+(x,v)$ the outward normal of the side of the hypercube C_i through which $L(x,v)$ enters C_i and n_i^- the outward normal of the side of the hypercube through which $L(x,v)$ leaves. Then

$$e\cdot\nabla_x l_i(x,v) = \frac{e\cdot n_i^+}{v\cdot n_i^+} - \frac{e\cdot n_i^-}{v\cdot n_i^-}. \tag{4.17}$$

Consequently this derivative is zero if the two sides are parallel and since $v\in S_0$,

$$\left|\sum_{i=1}^n \nabla_x l_i(x,v)\,\phi_i(x)\right| \le CKN. \tag{4.18}$$

Thanks to Estimate (4.18), we may deduce the result in L^p for $p > 2$ from the result in L^2. Indeed

$$\int_{B(0,2K)} \int_{v\in S_0} |\nabla_x T_N(x,v)|^p \, dv \, dx \le C(KN)^{p-2} \int_{x,v} |\nabla_x T_N(x,v)|^2 \, dv \, dx$$
$$\le C_K N^{p-2} \times N^{1-0} \, |E|,$$

if the result is true in L^2 for any $\theta < 1/2$. For $p < 2$, we may divide the integral in x, v in a domain where $|\nabla_x T_N| \ge 1$ and a domain where $|\nabla_x T_N| < 1$. The bound on the integral on the first domain is also a consequence of the estimate in L^2 and on the second domain

$$\int_{(x,v)\ s.t.\ |\nabla_x T_N|<1|} |\nabla_x T_N(x,v)|^p \, dv \, dx \le \int_{x,v} |\nabla T_N(x,v)|$$
$$\le CN|E|,$$

trivially, which gives the corresponding result since then $p - \theta p > 1$. Those two arguments are the precised equivalent of the interpolation argument we had previously.

It only remains to prove (4.16) with $p = 2$ and $\theta < 1/2$. This is a consequence of the almost orthogonality of the functions $\nabla_x l_i$. Since $v \cdot \nabla_x l_i = 0$, it is enough to do it for the first $d-1$ components $\partial_k l_i$ of $\nabla_x l_i$. We choose $k = 1$: the computation for any other $k \le d-1$ is the same because of the symmetry in S_0.

Let us compute the following. Take η a vector with $|\eta| \le 1/N$ and $\mathcal{N}_i$ is the set of j such that C_j intersects one of the half lines centered inside C_i and of direction inside S_0 (because of the definition of S_0, for any x, on a line connecting x, C_i and C_j, C_i is between x and C_j),

$$\Delta_i^\eta(t) = \sum_{j\in\mathcal{N}_i} \int_{S_0} \partial_{x_1} l_i(\eta+x_i+vt, v)\phi_i(\eta+x_i+vt)$$
$$\times \partial_{x_1} l_j(\eta+x_i+vt, v)\phi_j(\eta+x_i+vt)\, \psi(v)\, dv.$$

Fix $j \in \mathcal{N}_i$, a real t and a side of C_i, we denote by S_i the subspace of S_0 so that $L(x_0, v)$ enters C_i on the chosen side and therefore $\partial_{x_1} l_i$ is a constant. Then since $\nabla_{x_1} l_j$ is non zero as a function of v, on a space of measure $C(|x_i - x_j|\, N)^{1-d}$,

$$\left| \int_{S_0} \partial_{x_1} l_j(\eta + x_i - vt, v)\, \psi(v)\, dv \right| \le CN^{-d+1} \times |x_i - x_j|^{-d+1}.$$

But using the cancellations and provided ψ is a regular function, we can prove the better inequality

$$\left|\int_{S_0} \partial_{x_1} l_j(\eta + x_i - vt, v)\, \psi(v)\, dv\right| \leq CN^{-d} \times |x_i - x_j|^{-d}. \tag{4.19}$$

Denote by C_j^1 and C_j^2 the sides of C_j whose normal vectors n_j^1 and n_j^2 are parallel to e_1 and $\alpha_j^k(x,v)$ the function with value 1 if $L(x,v)$ intersects C_j^k. Note that since $v \in S_0$, there cannot exist $v, v' \in S_0$ such that $L(x,v)$ enters the hypercube on the side C_j^1 but $L(x,v')$ leaves the hypercube on C_j^2 or the converse. Therefore

$$\left|\int_{S_0} \partial_{x_1} l_j(\eta+x_i-vt, v)\, \psi\, dv\right| \leq \left|\int_{S_0} (\alpha_j^1(\eta+x_i-vt,v) - \alpha_j^2(\eta+x_i-vt,v))\, \frac{\psi}{v_1} dv\right|.$$

We know that $\alpha_j^2(x,v) = \alpha_j^1(x, R_{ij}v)$ with R_{ij} such that $|R_{ij}v - v| \leq C/N\, |x_i - x_j|$. Since the functions α_j^k are BV and $1/v_1$ is C^∞ over S_0, we immediately get (4.19) from the fact that α_j^k is positive on a subset of S_0 of diameter at most $C/(N\,|x_i - x_j|)$.
Now note that in $\Delta_i(t)$, in fact $\phi_i(\eta + x_i - vt)$ and $\phi_j(\eta + x_i - tv)$ are almost constant since $|\eta + x_i - vt|$ is equal to $t \pm 1/N$ and $|\eta + x_j - x_i + tv|$ to $|x_j - x_i| + t \pm 1/N$ (the points $x_i - tv$, x_i and x_j are almost on the same line if ∇l_j is not zero). So up to an approximation of the kind we already performed, we may take it constant and we then have thanks to (4.19)

$$\begin{aligned} |\Delta_i^\eta(t)| &\leq CN^{-d} t^{-s} \sum_{j \in \mathcal{N}_i} (|x_i - x_j| + t)^{-s}\, |x_i - x_j|^{-d} \\ &\leq CN^{-d} t^{-s} \sum_{k=1}^{N} (k/N + t)^{-s}\, (k/N)^{-d} \times k^{d-1}, \end{aligned}$$

summing first on all $j \in \mathcal{N}_i'$ which are at the same distance of x_i. Eventually we find

$$|\Delta_i^\eta(x)| \leq Ct^{-2s} \times \log N. \tag{4.20}$$

To conclude the proof, we note first that, with B_i the set of x such that $L(x,v)$ enters C_i on a given chosen side C_i^k, $k = 1 \ldots 2^d$,

$$\int_{B(0,2K)} \int_{S_0} \left|\sum_{i=1}^{n} \nabla_x l_i \phi_i(x)\right|^2 dv\, dx = 2^d \sum_{i=1}^{n} \sum_{j \in \mathcal{N}_i} \int_{S_0}\int_{B_i} \nabla_x l_i\, \phi_i\, \nabla_x l_j\, \phi_j\, dx\, dv.$$

Then we perform a change of variable from (x) to (η, t) where $t = |x - x_i|$ and $\eta + x_i$ is the point where $L(x, v)$ crosses the chosen side of C_i (thus $|\eta| \leq 1/N$) to get

$$\int_{B(0,2K)} \int_{S_0} \left| \sum_{i=1}^{n} \nabla_x l_i \phi_i(x) \right|^2 dv\, dx$$
$$\leq C \sum_{i,j=1}^{n} \int_{S_0} \int_{t \leq 2K} \int_{\eta \in C_i^k - x_i} (\nabla_x l_i\, \phi_i\, \nabla_x l_j\, \phi_j)(x_i - vt + \eta, v)\, \psi(\eta, t, v) d\eta\, dt\, dv.$$

Since ψ is a perfectly regular function, we may switch the order of integration and apply (4.20) to find

$$\int_{B(0,2K)} \int_{S_0} \left| \sum_{i=1}^{n} \nabla_x l_i \phi_i(x) \right|^2 dv\, dx \leq C\, \log N \sum_{i=1}^{n} \int_{t \leq 2K} \int_{\eta \in C_i^k - x_i} t^{-2s}\, d\eta\, dt$$
$$\leq C\, \log N \sum_{i=1}^{n} N^{1-d} \leq CN\, \log N\, |E|,$$

which finishes to prove (4.16) and the lemma. □

Proof of Prop. 4.5. The proof uses Lemma 4.2 and a standard approximation procedure.

Let us consider any nonnegative function f with compact support and which is constant on any hypercubes of the form $[i_1/N,\ i_1/N + 1/N] \times \ldots \times [i_d/N,\ i_d/N + 1/N]$ for a given integer N. Therefore f takes only a finite number of positive values $0 < \alpha_1 < \ldots < \alpha_n$. Denoting by E_i the set of points x where f is equal to α_i, we know that $E_i \in \mathcal{C}_N$ from the assumption on f. Hence for any $\theta < 1 - 1/\bar{p}$

$$\|T_s^* f\|_{L_v^p W_x^{\theta,p}} \leq \sum_{i=1}^{n} \alpha_i \|T_s^* \mathbb{I}_{E_i}\|_{L_v^p W_x^{\theta,p}} \leq C \sum_{i=1}^{n} \alpha_i |E|^{1/p}.$$

Denote by $f^*(t)$ the decreasing rearrangement corresponding to f (see [2]). Then $f^*(t)$ has value α_i on the interval $[\beta_{i+1},\ \beta_i]$ with $\beta_i = \sum_{j=i}^n |E_j|$. Consequently the Lorentz norm of f satisfies

$$\|f\|_{L^{p,1}} = \int_0^\infty t^{1/p} f^*(t) \frac{dt}{t} = \sum_{i=1}^{n} \alpha_i (\beta_i^{1/p} - \beta_{i+1}^{1/p}) \geq C \sum_{i=1}^{n} \alpha_i |E_i|^{1/p}.$$

So eventually we showed that for any $\theta < 1 - 1/\bar{p}$

$$\|T_s^* f\|_{L^p_v W^{\theta,p}_x} \le C\|f\|_{L^{p,1}}.$$

Since $L^{p,1}$ is embedded in $W^{-\alpha,p}$ for any $\alpha > 0$ and since we do not care about the critical case, this implies that for any $\theta < 1 - 1/\bar{p}$ and any function f as described at the beginning

$$\|T_s^* f\|_{L^p_v W^{\theta,p}_x} \le C\|f\|_{L^p}.$$

Now it is enough to note that functions with compact support and whose level sets belong to $\mathcal{C}_N$ for a given N, are dense in L^p for $p < \infty$ which concludes the proof of Prop. 4.5. $\square$

Appendix B: Hypoelliptic regularity on f

It was noticed recently in [4], that the operator $v \cdot \nabla_x$ has some regularizing effects of its own. More precisely in Theorem 1.1, the regularity gained on the average through additional derivatives in velocity on f or g, is also gained on f itself. Our purpose is not to investigate this kind of result and the theorem presented here is only a bit more general than the result of [4] but we wish to indicate briefly how one can obtain Bouchut's main result with our method.

Theorem 4.1 *Let f and g satisfy (1.1) and (1.4) with $\gamma \le 0$ and $1 < p_2,\ q_2 < \infty$. Then, for $L^{r_1,\infty}$ the Lorentz space of parameters r_1 and ∞,*

$$\|f\|_{L^{r_1,\infty}_v(\dot{B}^{s,r_2}_{\infty,\infty})} \le C\|g\|^{\theta}_{W^{\gamma,q_1}_v(L^{q_2}_x)} \times \|f\|_{W^{\beta,p_1}_v(L^{p_2}_x)},$$

with

$$\begin{aligned} &\frac{1}{r_i} = \frac{\theta}{q_i} + \frac{1-\theta}{p_i} \quad i = 1,\,2, \qquad s = (1-\alpha)\theta, \\ &\theta = \frac{\beta}{1+\beta-\gamma}. \end{aligned} \tag{4.21}$$

Remarks.
1. Just as in [4], we are unable to treat correctly the case $\gamma > 0$ except when both f and g belong to L^2. The correct regularity should be obtained just

by extending the formula of the theorem, however our method gives a lower regularity.
2. Theorem 4.1 is not really much better than the corresponding result of [4]. Its only advantage is that in [4], f and g had to belong to the same $L^p_{x,v}$ but it gives f in a modified Besov space instead of the Sobolev space of same homogeneity.

Proof. We use the same basic idea as for the proof of Theorem 1.1 and hence we will not give all details.
We decompose f and g into dyadic annulus in the Fourier space in x, thus obtaining two sequences f_k and g_k where k is the indice of the annulus (*i.e.* 2^k is the order of one derivative in x). Of course

$$v \cdot \nabla_x f_k = g_k.$$

We again consider for λ_k to be fixed later

$$(v \cdot \nabla_x + \lambda_k) f_k = g_k + \lambda_k f_k.$$

Hence we obtain

$$f_k(x,v) = S_k g_k + \lambda_k S_k f_k,$$

with

$$S_k h(x,v) = \int_0^\infty h(x - vt,\, v)\; e^{-\lambda_k t}\, dt.$$

We consider K_ϵ a regularizing kernel in velocity and we write

$$f_k(x,v) = (f_k - K_\epsilon \star_v f_k) + K_\epsilon \star_v (S_k g_k) + \lambda_k K_\epsilon \star_v (S_k f_k).$$

Now of course if

$$g = \partial_{v_i} h(x,v),$$

then

$$S_k g_k = \partial_{v_i} S_k h_k + \int_0^\infty t \partial_{x_i} h_k(x - vt,\, v) e^{-\lambda_k t} dt.$$

Hence for an arbitrary $\gamma \leq 0$

$$\|K_\epsilon \star_v S_k g_k\|_{L^{q_1}_v(L^{q_2}_x)} \leq \lambda_k^{-1}\, 2^{k\alpha}(1 + \lambda_k^\gamma 2^{-k\gamma} + \epsilon^\gamma) \times \|g_k\|_{W^{\gamma,q_1}_v(L^{q_2}_x)}.$$

As to $S_k f_k$, we have

$$\partial_{x_i} S_k f_k = \int_0^\infty \frac{e^{-\lambda_k t}}{t} (\partial_{v_i} f)(x - vt,\, v) - \partial_{v_i} \int_0^\infty \frac{e^{-\lambda_k t}}{t} f(x - vt,\, v).$$

Here we may face the same problem of integrability in t as in the proof of Theorem 1.1. We refer the reader to the end of the corresponding proof for the way to treat it. Notwithstanding that, we obtain, for $\beta \in \mathbb{N}$ the final result being just the interpolation between integer values of β

$$K_\epsilon \star_v (S_k f_k) = F_k^1 + F_k^2,$$

with

$$\|F_k^1\|_{L_v^{p_1}(L_x^{p_2})} \leq 2^{-k\beta}\ \lambda_k^\beta \times \|f_k\|_{W_v^{\beta,\,p_1}(L_x^{p_2})},$$
$$\|F_k^2\|_{L_v^{r_1,\infty}(B_{\infty,\,x}^{s,r_2})} \leq 2^{-k\beta}\ \lambda_k^\beta\ \epsilon^{-\beta} \times \|f_k\|_{L_v^{r_1,\infty}(B_{\infty,\,x}^{s,r_2})}.$$

Eventually

$$\|K_\epsilon \star_v f_k - f_k\|_{L_v^{p_1}(L_x^{p_2})} \leq \epsilon^\beta \times \|f_k\|_{W_v^{\beta,\,p_1}(L_x^{p_2})}.$$

We minimize in λ_k and ϵ and take

$$\lambda_k = \mu_k\ 2^{k-k(1-\alpha)/(1+\beta-\gamma)}, \quad \epsilon = \frac{1}{2}\mu_k\ 2^{-k(1-\alpha)/(1+\beta-\gamma)}.$$

With these values, we know that

$$\|F_k^1\|_{L_v^{p_1}(\dot W_x^{s,p_2})} \leq \mu_k\ \mu_k^\beta \times \|f_k\|_{W_v^{\beta,\,p_1}(L_x^{p_2})},$$
$$\|K_\epsilon \star_v S_k g_k\|_{L_v^{q_1}(\dot W_x^{s,q_2})} \leq \mu_k^{\gamma-1} \times \|g_k\|_{W_v^{\gamma,\,q_1}(L_x^{q_2})}.$$

We use μ_k to interpolate between $L_v^{p_1}(\dot W_x^{s,p_2})$ and $L_v^{q_1}(\dot W_x^{s,q_2})$, we eventually find

$$\|f_k\|_{L_v^{r_1,\infty}(B_{\infty,\,x}^{s,r_2})} \leq C\|g_k\|^\theta_{W_v^{\gamma,\,q_1}(L_x^{q_2})} \times \|f_k\|^{1-\theta}_{W_v^{\beta,\,p_1}(L_x^{p_2})} + \frac{1}{2}\|f_k\|_{L_v^{r_1,\infty}(B_{\infty,\,x}^{s,r_2})}.$$

It is now enough to sum on k to conclude.

If $\gamma > 0$, this method fails, one has to work in L^2 with $\beta \geq \gamma$ and $\beta \geq 1-\gamma$ and use a duality method based on the identity

$$\partial_{x_i} S_k f_k = S_k \partial_{v_i} g_k + \lambda_k S_k \partial_{v_i} f_k - \partial_{v_i} f_k.$$

Acknowledgement.
The authors would like to thank B. Perthame and F. Bouchut for enlightening conversations on the subject.

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