Macroscopic limit of Vlasov type equations with friction

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Abstract. The purpose of this paper is to investigate the limit of some kinetic equations with a strong force. Due to friction, the solution concentrates to a monokinetic distribution so as to keep the total of force bounded and in the limit we recover a macroscopic system. This kind of asymptotics is a natural question when the mass of the particles is very small or their inertia is neglected. After that we also study the properties of the limit system and especially the uniqueness of solutions which provides the full convergence of the family of solutions to the kinetic equation.

Résumé. Cet article se propose d'étudier la limite de solutions d'une équation cinétique avec frottement lorsque les termes de force deviennent prédominants. A cause du frottement, les solutions se concentrent progressivement en vitesse de manière à ce que la somme des forces reste bornée ; à la limite cette concentration nous oblige à remplacer l'équation cinétique par un système macroscopique. Cette problème apparait notamment quand on fait tendre vers zéro la masse des particules ou quand on néglige leur inertie. Enfin certaines propriétés du système, et particulièrement l'unicité, seront détaillées afin d'obtenir une convergence de toute la suite des solutions et pas seulement d'une suite extraite.

Introduction

We are interested in the behaviour when ϵ vanishes of kinetic equations of the kind

$$(1)\qquad \begin{cases} \dfrac{\partial f_\epsilon}{\partial t} + v\cdot\nabla_x f_\epsilon + \dfrac{1}{\epsilon}\mathrm{div}_v\big((F[f_\epsilon]-v)f_\epsilon\big) \;=\; 0\;, \quad t\geq 0,\ (x,v)\in \mathbb{R}^{2\mathrm{d}}, \\ f_\epsilon(t=0,x,v) \;=\; f^0(x,v)\;. \end{cases}$$

Here we work in any dimension d and the force term $F[f_\epsilon]$ only depends on the mass density ρ_ϵ or on the first moment j_ϵ of $f\epsilon$, defined by

$$(2)\qquad \begin{cases} \rho_\epsilon(t,x) &= \displaystyle\int_{\mathbb{R}^d} f_\epsilon(t,x,v)dv \ , \\ j_\epsilon(t,x) &= \displaystyle\int_{\mathbb{R}^d} v f_\epsilon(t,x,v)dv \ . \end{cases}$$

The guiding example throughout this paper will be the modified Vlasov Stokes system

$$(3)\qquad \begin{cases} \dfrac{\partial f_\epsilon}{\partial t} + v\cdot\nabla_x f_\epsilon + \dfrac{1}{\epsilon}\mathrm{div}_v\big((g + K\star_x j_\epsilon - v)f_\epsilon\big) \ = \ 0 \ , \quad t\geq 0, \ (x,v)\in\mathbb{R}^6, \\ f_\epsilon(t=0,x,v) \ = \ f^0(x,v) \ . \end{cases}$$

While the general equation (1) contains for instance the classical Vlasov-Poisson system, the system (3) is a simplified model for the dynamics of dilute particles in a Stokes flow and submitted to gravity (g in the above equation) when we take for the matrix K

$$(4)\qquad K(x) \ = \ -c\left(\frac{Id}{|x|} + \frac{x\otimes x}{|x|^3}\right) \ , \quad c>0 \ .$$

This model is derived by P-E Jabin and B. Perthame in [15] and its basic properties are stated in [14] and in [6] by I. Gasser, P-E Jabin and B. Perthame. K. Hamdache also worked on the existence for a slightly different model in [11]. In another situation, kinetic equations for a system of particles in a potential flow have been introduced by D. Herrero, B. Lucquin-Desreux and B. Perthame in [12] and by G. Russo and P. Smeraka in [17]. An evolution equation, close to the limit systems we obtain here (see (5) and (6) below), has also been derived for an infinite suspension of particles (see J. Rubinstein and J.B. Keller in [16]).

Formally, when ϵ converges to zero, the limits of the systems (1) is

$$(5)\qquad \begin{cases} \dfrac{\partial \rho}{\partial t} + \mathrm{div} j \ = \ 0 \ , \\ j(t,x) \ = \ \rho F[\rho, j] \ , \\ \rho(t=0,x) \ = \ \rho^0(x) \ , \end{cases}$$

and in the special case of (3), we obtain

$$(6)\qquad \begin{cases} \dfrac{\partial \rho}{\partial t} + \mathrm{div} j \ = \ 0 \ , \\ j(t,x) \ = \ \rho(K\star_x j + g) \ . \end{cases}$$

This paper aims at proving this limit rigorously, for some F or K regular enough and in particular more regular than the matrix K defined by (4). This additional regularity is necessary because in these limits, f_ϵ concentrates to a monokinetic distribution $\rho(x)\delta(v - u)$ with $\rho u = j$. Notice that the initial data is given and does not depend on ϵ, the concentration is thus only due to the natural evolution of the equation.

This kind of singular perturbation and this way of deriving macroscopic limits is quite recent and the usual methods, see for instance R.T. Glassey in [7], do not apply. The main difficulty is that the density function f, in this limit, concentrates as a Dirac mass. This problem is related to the quasi-neutral limit for plasmas (see Y. Brenier [3], and E. Grenier [9]) or the limit of the Vlasov-Poisson system towards the pressureless Euler-Poisson system (see V. Sandor [18]) where the same phenomenon of concentration occurs. A remarkable difference is however that here the only scaling of the force term is enough, and it is not necessary to also scale the initial data. Another remarkable feature is that our limiting system does not have a notion of dissipative solution and the method of [3] cannot be used here. Another class of singular perturbation problems with strong force terms has been treated by E. Frnod and E. Sonnendrcker in [5] (fixed magnetic force) and F. Golse and L. Saint-Raymond in [8] (the so-called gyrokinetic limit). Keeping L^p $(p > 1)$ bounds on f, the analysis in [5] is then based on two-scale Young measures (see G. Allaire [1] and G. N'Guetseng [10]), and in [8] it is based on a compactness argument due to Delort. The methods developed in these papers cannot be applied here because of the concentration phenomenon and the different structure of the force term.

We will first present three theorems : the first one proves the limit for "regular" force terms, the second gives some properties of existence and uniqueness for the macroscopic system with the same regularity assumption, whereas the last one studies the system (6). The rest of the paper will be devoted to the proof of these theorems.

1. Main results

We will only deal with force terms $F[\rho_\epsilon, j_\epsilon]$ which are a sum of two convolution operators in ρ_ϵ and j_ϵ

$$F[\rho_\epsilon, j_\epsilon] \;=\; A \star_x \rho_\epsilon + K \star_x j_\epsilon + G(x) \;, \tag{7}$$

with the assumption

$$A, G \in (C_0(\mathrm{I\!R}^{\mathrm{d}}))^{\mathrm{d}}, \quad \mathrm{K} \in (\mathrm{C}_0(\mathrm{I\!R}^{\mathrm{d}}))^{\mathrm{d}^2} \;. \tag{8}$$

Alternatively, we will aslo use the assumption of the negativity of the operator K, more precisely

$$\int_{\mathrm{I\!R}^{\mathrm{d}}} u(x) \cdot (K \star_x u(x)) \;\leq\; 0 \;, \quad \forall u \in (C_0(\mathrm{I\!R}^{\mathrm{d}}))^{\mathrm{d}} \;. \tag{9}$$

We consider equation (1) for a non-negative bounded initial data with finite mass and finite kinetic energy

$$(10)\qquad \begin{cases} f^0 \ \in \ L^1 \cap L^\infty(\mathbb{R}^{2\mathrm{d}}) \ , \\ E^0 \ = \ \displaystyle\int_{\mathbb{R}^{2\mathrm{d}}} |v|^2 f^0(t,x,v)dxdv \ < \ +\infty \ . \end{cases}$$

For developments we will also use the functional

$$(11)\qquad \Delta_\epsilon(t) \ = \ \int_{\mathbb{R}^{2\mathrm{d}}} |v - F[\rho_\epsilon, j_\epsilon]|^2 f_\epsilon(t,x,v)dxdv \ .$$

The systems (1) admit the following a priori estimates

$$(12)\qquad \|f_\epsilon(t,.,.)\|_{L^1(\mathbb{R}^{2\mathrm{d}})} \ = \ \|f^0\|_{L^1(\mathbb{R}^{2\mathrm{d}})} \ ,$$

$$(13)\qquad \|f_\epsilon(t,.,.)\|_{L^\infty(\mathbb{R}^{2\mathrm{d}})} \ \le \ e^{\frac{dt}{\epsilon}} \|f^0\|_{L^\infty(\mathbb{R}^{2\mathrm{d}})} \ .$$

We will consider weak solutions of equation (1), which we define as distributional solutions to (1) satisfying the natural conditions

$$(14)\qquad \begin{cases} f_\epsilon \in L^\infty([0,\ T], L^1 \cap L^\infty(\mathbb{R}^{2\mathrm{d}})) \quad \forall \mathrm{T} \ge 0, \\ f_\epsilon \in C([0,\ \infty\,[\ , L^1(\mathbb{R}^{2\mathrm{d}})), \\ E_\epsilon(t) \ = \ \displaystyle\int_{\mathbb{R}^{2\mathrm{d}}} |v|^2 f_\epsilon(t,x,v)dxdv \ < \infty \quad \forall t \ge 0, \\ f_\epsilon \ \textit{is the weak* limit in } L^\infty([0,\ T], L^p(\mathbb{R}^{2\mathrm{d}})) \\ \textit{of classical solutions to } (1) \textit{ for all } 1 < p \le \infty. \end{cases}$$

Notice that the energy and L^1 estimates in (14) give estimates in L^1 for ρ_ϵ and j_ϵ. We are thus able to give a precise meaning to (1) in the space of distributions for weak solutions since $F[\rho_\epsilon, j_\epsilon] \in L^1_{loc}$. They also satisfy the a priori estimates (12) and (13). For a precise theory of existence and estimates of this kind of systems, see A. Arsenev [2], R.J. DiPerna and P.L. Lions [4], and also [6], [7], E. Horst [13] and [14].

In all the paper, M^1 will denote the space of Radon measures. We are now ready to state our three theorems

Theorem 1 : *Let* (f_ϵ) *be a sequence of weak solutions in the sense of* (14) *to equation* (1). *Assume* (7), (8) *and that the kinetic energy* $E_\epsilon(t)$ *is uniformly bounded over any interval* $[0,\ T]$. *Then, as* ϵ *converges to zero, there is a subsequence such that*

(i) $\rho_\epsilon \rightharpoonup \bar\rho(x)$, $j_\epsilon \rightharpoonup \bar\jmath(x)$ *weakly in* $L^\infty([0,\ T], M^1(\mathbb{R}^{\mathrm{d}}))$, *and* $\bar\rho$ *and* $\bar\jmath$ *are solutions to the system* (5),

(ii) *if* $\|K\|_{L^\infty}\|f^0\|_{L^1} < 1$ *or if* (9) *holds, then* $\Delta_\epsilon(t) \to 0$ *in* $C([t^\star, \infty\,[)$ *for all* $t^\star > 0$.

(iii) *if* $K,\ A,\ G$ *are in* $W^{1,\infty}$ *and* (9) *holds, then there exists a constant* $C > 0$ *such that* $\Delta_\epsilon(t) \le C(t^\star)\epsilon^2 \quad \forall t \in [t^\star, \infty\,[\,,\ t^\star > 0$ *and*
$f_\epsilon \rightharpoonup \bar\rho(x)\delta(v - F[\bar\rho, \bar\jmath])$ *weakly in* $L^\infty([0,\ T], M^1(\mathbb{R}^{2d}))$.
Moreover if condition (9) *holds true, the kinetic energy is bounded.*

Remarks.

1.The main limitation of this theorem comes from assumption (8), because it does not allow the natural singularity in the forces. This is due to the fact that we only have M^1 estimates in the phase space for f_ϵ, ρ_ϵ or j_ϵ and we need to pass to the limit in the term $\rho_\epsilon F[\rho_\epsilon, j_\epsilon]$. However, if we suppose that $\mathrm{div}F[\bar\rho, \bar\jmath] = 0$ (again a natural condition in view of (4)), then the limit system (5) conserves all L^p norm of $\bar\rho$ and of $\bar\jmath$ because we then have

$$\frac{\partial \bar\rho}{\partial t} + F[\bar\rho, \bar\jmath] \cdot \nabla \bar\rho \ = \ 0 \ ,$$

while the second equation of (5) shows that the L^p norm $\bar\jmath$ is dominated by the L^p norms of $\bar\rho$, (see the theorem 3 below). Since it is very natural to get $\mathrm{div}F[\bar\rho, \bar\jmath] = 0$, especially in the case of Vlasov-Stokes equation ($A = 0$, $G = cst$, $\mathrm{div}K = 0$), it is a natural open question to know if condition (8) can be removed in that case.

2. A crucial step in this theorem is the estimate for the functional $\Delta_\epsilon(t)$ and the main difficulty is to prove that it converges to zero even if K, A or G are not differentiable and without condition (9). If K, A and G are C^1 and if (9) is true, we prove that Δ_ϵ is less than ϵ^2. Notice nevertheless that, even in this case, the decay is not enough to compensate the exponential growth of f_ϵ and the question of the regularity in x of ρ_ϵ or j_ϵ is still not solved.

3. From the first remark we can expect that $\bar\rho$ and $\bar\jmath$ have the same regularity as ρ^0 and j^0 ($L^1 \cap L^{\frac{5}{3}}$ for $\bar\rho$ in dimension three for example). Another natural question arising from theorem 1 would hence be to prove weak L^1 convergence instead of weak measure convergence.

Theorem 2 : *Let* F *be given by* (7)*, with conditions* (8) *and* (9)*, and assume that* div K, div A *and* ∇G *belong* L^∞*. Then*
(i) *the second equation of the system* (5) *determines uniquely* j *in* L^1 *as a function of* ρ *in* L^1*,*
(ii) *the system has distributional solutions* $\rho,\ j \in L^\infty([0,\ T],\ L^1(\mathbb{R}^d))$*,* $\forall T > 0$ *for any non-negative* ρ^0 *in* L^1*,*
(iii) *if* ρ^0 *belongs to* $W^{1,1}$*, and is small in* L^1 *the solution is unique for small times,*
(iv) *if* K *and* $A \in W^{1,\infty}$*, uniqueness holds globally in time.*

This theorem provides a framework for existence of solutions to the system (5) also. But a more general existence framework is easy to settle. It should also be noticed that theorem 1 already provides a partial existence result (at least for ρ^0 and j^0 given by the second equation being the zeroth and the first moment in velocity of a function in $L^1 \cap L^\infty$).

However existence can be proved more generally for $\rho^0 \in L^1$. We can still get a stronger result (no smallness assumption) if we precise the structure of the matrix K. It shows that we can allow a singularity in the matrix whereas we are unable to prove a variant of theorem 1 with any singularity.

Theorem 3 : *Assume $d = 3$, F is given by (7) with $A = 0$, $G \in W^{1,\infty}$ and K the matrix given by (4). Then, the system (5) has distributional solutions ρ, $j \in L^\infty([0,\ T],\ L^1(\mathbb{R}^3))$ for any initial data ρ^0 in $L^1 \cap L^{\frac{3}{2}}$. If ρ^0 belongs to $W^{1,3}$, this solution is unique locally in time.*

2. Proof of theorem 1

This proof is divided into four parts. First of all we show that the main quantities have limits and we explain why $\bar\rho$ and $\bar\jmath$ satisfy the system (5) (part (i) of the theorem), then we prove that the functional Δ_ϵ defined by (11) converges to zero (part (ii) of the theorem), the next subsection being devoted to the case A, K, $G \in W^{1,\infty}$ and the Dirac form of the limit of f_ϵ. At last we explain why condition (9) ensures the uniform boundedness of the kinetic energy.

2.1 Existence of limits for f_ϵ, ρ_ϵ, j_ϵ and $\rho_\epsilon F[\rho_\epsilon, j_\epsilon]$

We prove the point (i). First of all the conservative form of equation (1) and condition (14) imply the estimate (12). We therefore have

$$\|\rho_\epsilon\|_{L^\infty([0,\ T],L^1(\mathbb{R}^d))} = \|f^0\|_{L^1(\mathbb{R}^{2d})}\ . \tag{15}$$

$$\|j_\epsilon\|_{L^\infty([0,\ T],L^1(\mathbb{R}^d))} \le \|f^0\|^{\frac{1}{2}}_{L^1(\mathbb{R}^{2d})} \sup E_\epsilon^{\frac{1}{2}}(t). \tag{16}$$

Therefore, we can extract a subsequence so as to get

$$\begin{aligned} f_\epsilon &\rightharpoonup \bar f \quad \textit{weakly in } L^\infty([0,\ T], M^1(\mathbb{R}^{2d})),\\ \rho_\epsilon &\rightharpoonup \bar\rho \quad \textit{weakly in } L^\infty([0,\ T], M^1(\mathbb{R}^{d})),\\ j_\epsilon &\rightharpoonup \bar\jmath \quad \textit{weakly in } L^\infty([0,\ T], M^1(\mathbb{R}^{d})),\end{aligned}$$

The continuity equation $\frac{\partial\rho_\epsilon}{\partial t} + \mathrm{div} j_\epsilon = 0$ is obviously satisfied thanks to the boundedness of the force term $F[\rho_\epsilon, j_\epsilon]$ (assumption (8)). To show that $\bar\rho$ and $\bar\jmath$ are solutions to the system (5) and end the proof of the first part of theorem 1, we only need to apply lemmas 1 and 2.

Lemma 1 : *Assume ρ_ϵ and j_ϵ are two sequences uniformly bounded in $L^\infty([0,\ T],\ L^1(\mathbb{R}^d))$, weakly converging in $L^\infty([0,\ T], M^1(\mathbb{R}^d))$ and satisfying the continuity equation. Then the product $\rho_\epsilon F[\rho_\epsilon, j_\epsilon]$ converges weakly in $L^\infty([0,\ T], M^1(\mathbb{R}^d))$ towards $\bar\rho F[\bar\rho, \bar\jmath]$.*

Proof of lemma 1

Notice first that, using assumption (8), $F[\bar\rho,\bar\jmath]$ belongs to $L^\infty([0,\ T],C_0(\mathbb{R}^d))$ and therefore $\bar\rho F[\bar\rho,\bar\jmath]$ is well defined.

Formula (7) allows us to decompose $\rho_\epsilon F[\rho_\epsilon,j_\epsilon]$ in three terms

$$\rho_\epsilon F[\rho_\epsilon,j_\epsilon] \;=\; \rho_\epsilon(K\star j_\epsilon)+\rho_\epsilon A\star\rho_\epsilon+\rho_\epsilon G(x).$$

We will show that the three terms converges weakly in $L^\infty([0,\ T],C_0(\mathbb{R}^d))$. This is obvious for $\rho_\epsilon G(x)$. The only problem is to get some time compactness in the two other products, which is done by using the continuity equation. This is well known for $\rho_\epsilon A\star\rho_\epsilon$ and so we only explain the procedure for the first term since $\partial_t j_\epsilon$ is not a priori uniformly in any negative Sobolev space.

The term $\rho_\epsilon(K\star j_\epsilon)$ is bounded in $L^\infty([0,\ T],L^1(\mathbb{R}^d))$ and so, extracting a subsequence, it converges weakly in $L^\infty([0,\ T],M^1(\mathbb{R}^d))$. We only have to identify its limit. Let us first choose a regularization K_δ of K in $C^1(\mathbb{R}^d)$. For all $\phi(t,x)$ in $C^1_0(\mathbb{R}^{d+1})$, we have

$$\int_{\mathbb{R}^d}\phi(t,x)\rho_\epsilon(t,x)(K_\delta\star j_\epsilon)dx \;=\; \int_{\mathbb{R}^d} j_\epsilon(t,x)(\tilde K_\delta\star\rho_\epsilon\phi)dx\ ,$$

with, if K^T denotes the transpose of the matrix K,

$$\tilde K_\delta(x) \;=\; K_\delta^T(-x).$$

The continuity equation and the L^1 bound on j_ϵ imply that ρ_ϵ belongs to $W^{1,\infty}([0,\ T],$ $W^{-1,1}(\mathbb{R}^d)$. Applying Ascoli theorem, this shows that $\tilde K_\delta\star\phi\rho_\epsilon$ converges strongly in $C_0([0,\ T],\ C_0(\mathbb{R}^d))$ towards $\tilde K_\delta\star\phi\bar\rho$, which enables us to conclude.

□

Lemma 2 : *The limit of $\rho_\epsilon F[\rho_\epsilon,j_\epsilon]$ in $w-L^\infty([0,\ T],M^1(\mathbb{R}^d))$ is precisely the limit $\bar\jmath$ of j_ϵ in the same space.*

Proof of lemma 2

Multiplying the equation (1) by v and integrating in velocity, we obtain

$$\begin{aligned}&\frac{\partial j_\epsilon}{\partial t}+\mathrm{div}_x\mathcal{E}_\epsilon+\frac{1}{\epsilon}(j_\epsilon-\rho_\epsilon F[\rho_\epsilon,j_\epsilon]) \;=\; 0\ ,\\ &\mathcal{E}_\epsilon(t,x) \;=\; \int_{\mathbb{R}^d} v\otimes v f_\epsilon(t,x,v)dv\ .\end{aligned}\tag{17}$$

The uniform bound on the kinetic energy gives a uniform bound on $\mathcal{E}_\epsilon$ in $L^\infty([0,\ T],$ $L^1(\mathbb{R}^d))$. As a consequence we immediately deduce that $j_\epsilon-\rho_\epsilon F[\rho_\epsilon,j_\epsilon]$ converges towards zero in $W^{-1,1}([0,\ T]\times\mathbb{R}^d))$.

Since we already know that j_ϵ and $\rho_\epsilon F[\rho_\epsilon,j_\epsilon]$ converge towards $\bar\jmath$ and $\bar\rho F[\bar\rho,\bar\jmath]$ in $w-L^\infty([0,\ T],M^1(\mathbb{R}^d))$, the two limits are necessarily equal.

□

2.2 Concentration in velocity

First of all, let us consider sequences of regularisations K_δ, A_δ and G_δ in $C_0^1(\mathbb{R}^d)$ of K, A and G. Using these sequences, we define a force term F_δ and the functional

$$\Delta_{\epsilon,\delta} \;=\; \int_{\mathbb{R}^{2d}} |v - F_\delta[\rho_\epsilon, j_\epsilon]|^2 f_\epsilon(t,x,v)\; dxdv\;. \tag{18}$$

We are able to prove lemma 3 which almost directly implies that Δ_ϵ vanishes with ϵ (thus ending the proof of part (ii) of the theorem), since $\Delta_{\epsilon,\delta}$ converges to Δ_ϵ with δ, uniformly in ϵ.

Lemma 3 : *if* $\|K_\delta\|_{L^\infty}\|f^0\|_{L^1} < 1$, then $\Delta_{\epsilon,\delta} \;=\; \alpha(\epsilon,\delta) + \beta(\delta)$, *on* $[t^\star,\ T]$ *for all* $t^\star > 0$, *with* α *a function vanishing with* ϵ *for* δ *fixed and* β *a function vanishing with* δ.

Proof of lemma 3

$$\begin{aligned}\frac{d}{dt}\Delta_{\epsilon,\delta} \;&=\; -2\int_{\mathbb{R}^{2d}} (v - F_\delta[\rho_\epsilon, j_\epsilon])\cdot(K_\delta\star\partial_t j_\epsilon) f_\epsilon(t,x,v)\; dxdv\\ &\quad -2\int_{\mathbb{R}^{2d}} (v - F_\delta[\rho_\epsilon, j_\epsilon])\cdot(A_\delta\star\partial_t \rho_\epsilon) f_\epsilon(t,x,v)\; dxdv\\ &\quad +\int_{\mathbb{R}^{2d}} |v - F_\delta[\rho_\epsilon, j_\epsilon]|^2 \partial_t f_\epsilon(t,x,v)\; dxdv \;=\; \mathrm{I}+\mathrm{II}+\mathrm{III}\;.\end{aligned}$$

Let us first deal with I. Using equation (17) on j_ϵ, we find

$$\begin{aligned}\mathrm{I} \;&=\; 2\int_{\mathbb{R}^{2d}} (v - F_\delta[\rho_\epsilon, j_\epsilon])\cdot\Big(K_\delta\star\big(\mathrm{div}_x(\mathcal{E}) + \frac{1}{\epsilon}(j_\epsilon - \rho_\epsilon F[\rho_\epsilon, j_\epsilon])\big)\Big) f_\epsilon(t,x,v)dxdv\\ &=\; 2\int_{\mathbb{R}^{2d}} (v - F_\delta[\rho_\epsilon, j_\epsilon])\cdot(\nabla K_\delta\star\mathcal{E}) f_\epsilon(t,x,v)dxdv\\ &\quad +\frac{2}{\epsilon}\int_{\mathbb{R}^{2d}} (j_\epsilon - \rho_\epsilon F_\delta[\rho_\epsilon, j_\epsilon])\cdot(K_\delta\star(j_\epsilon - \rho_\epsilon F[\rho_\epsilon, j_\epsilon]))dxdv\;.\end{aligned}$$

Thus, setting $c = \|K_\delta\|_{L^\infty}\|f^0\|_{L^1}$,

$$\begin{aligned}\mathrm{I} \;&\le\; 2\Delta_{\epsilon,\delta}^{\frac12}\times\|f^0\|_{L^1}^{\frac12}\times\|\nabla K_\delta\|_{L^\infty}\times\|\mathcal{E}\|_{L^1} + \frac{2c}{\epsilon}\Delta_{\epsilon,\delta}\\ &\quad +\frac{2}{\epsilon}\int_{\mathbb{R}^6} (j_\epsilon - \rho_\epsilon F_\delta[\rho_\epsilon, j_\epsilon])\cdot(K_\delta\star(\rho_\epsilon F_\delta[\rho_\epsilon, j_\epsilon] - \rho_\epsilon F[\rho_\epsilon, j_\epsilon]))dxdv\\ &\le\; \left(C(\delta) + \frac{\gamma(\delta)}{\epsilon}\right)\Delta_{\epsilon,\delta}^{\frac12} + \frac{2c}{\epsilon}\Delta_{\epsilon,\delta}\;,\end{aligned}$$

with $C(\delta)$ bounded for δ fixed (it depends on $\|\nabla K_\delta\|_{L^\infty}$) and $\gamma(\delta)\to 0$ as $\delta\to 0$. Notice that if condition (9) holds we can take c equal to zero.

For II, we use the continuity equation

$$\begin{aligned}
\mathrm{II} &= 2\int_{\mathbb{R}^{2d}} (v - F_\delta[\rho_\epsilon, j_\epsilon]) \cdot (A_\delta \star \mathrm{div} j_\epsilon) f_\epsilon(t,x,v)\, dxdv \\
&= -2\int_{\mathbb{R}^{2d}} (v - F_\delta[\rho_\epsilon, j_\epsilon]) \cdot (\nabla A_\delta \star j_\epsilon) f_\epsilon(t,x,v)\, dxdv \\
&\le C(\delta)\Delta_{\epsilon,\delta}^{\frac{1}{2}} .
\end{aligned}$$

And for III, we insert equation (1) on f_ϵ

$$\begin{aligned}
\mathrm{III} &= -\int_{\mathbb{R}^{2d}} |v - F_\delta[\rho_\epsilon, j_\epsilon]|^2 \mathrm{div}_x(v f_\epsilon) dxdv \\
&\quad - \frac{1}{\epsilon}\int_{\mathbb{R}^{2d}} |v - F_\delta[\rho_\epsilon, j_\epsilon]|^2 \mathrm{div}_v((F[\rho_\epsilon, j_\epsilon] - v) f_\epsilon) dxdv \\
&= -2\int_{\mathbb{R}^{2d}} \big((v\cdot\nabla_x) F_\delta[\rho_\epsilon, j_\epsilon]\big) \cdot (v - F_\delta[\rho_\epsilon, j_\epsilon]) f_\epsilon dxdv \\
&\quad + \frac{2}{\epsilon}\int_{\mathbb{R}^{2d}} (v - F_\delta[\rho_\epsilon, j_\epsilon]) \cdot (F[\rho_\epsilon, j_\epsilon] - v) f_\epsilon dxdv \\
&= -2\int_{\mathbb{R}^{2d}} \big((v\cdot\nabla_x) F_\delta[\rho_\epsilon, j_\epsilon]\big) \cdot (v - F_\delta[\rho_\epsilon, j_\epsilon]) f_\epsilon dxdv \\
&\quad - \frac{2}{\epsilon}\Delta_{\epsilon,\delta} + \frac{2}{\epsilon}\int_{\mathbb{R}^{2d}} (v - F_\delta[\rho_\epsilon, j_\epsilon]) \cdot (F[\rho_\epsilon, j_\epsilon] - F_\delta[\rho_\epsilon, j_\epsilon]) f_\epsilon dxdv ,
\end{aligned}$$

which shows that

$$\mathrm{III} \le C(\delta)\Delta_{\epsilon,\delta}^{\frac{1}{2}} - \frac{2}{\epsilon}\Delta_{\epsilon,\delta} + \frac{\gamma(\delta)}{\epsilon}\Delta_{\epsilon,\delta}^{\frac{1}{2}} .$$

Putting all this together, we get

$$\frac{d}{dt}\Delta_{\epsilon,\delta} \le \left(C(\delta) + \frac{\gamma(\delta)}{\epsilon}\right)\Delta_{\epsilon,\delta}^{\frac{1}{2}} - \frac{2(1-c)}{\epsilon}\Delta_{\epsilon,\delta} .$$

We recall here that $c = \|K_\delta\|_{L^\infty}\|f^0\|_{L^1}$ ($c = 0$ if (9) is true) and is strictly less than 1. Using Gronwall lemma, we eventually end up with

$$\Delta_{\epsilon,\delta}(t) \le \mathrm{Max}\left(e^{-\frac{(1-c)t}{\epsilon}}\Delta_{\epsilon,\delta}(0), \quad \frac{\epsilon^2}{(1-c)^2}C^2(\delta) + \frac{\gamma(\delta)}{(1-c)^2}\right) ,$$

which proves the lemma.

□

2.3 A simpler case

We now prove (iii). When K, A and G belong to $W^{1,\infty}$ and condition (9) is true, then minor modifications of the above proof show that

$$\Delta_\epsilon(t) \ \leq \text{Max}\left(e^{-\frac{2t}{\epsilon}}\Delta_\epsilon(0), \quad C\epsilon^2\right) \ ,$$

indeed, in the calculation of section 2.2 we do not regularize with δ and the main difficult term $\int(j_\epsilon - \rho_\epsilon F[\rho_\epsilon, j_\epsilon]) \cdot K \star (j_\epsilon - \rho_\epsilon F[\rho_\epsilon, j_\epsilon])$ is negative. So we obtain the inequalities

$$I \leq C\Delta_\epsilon^{\frac{1}{2}}, \quad II \leq C\Delta_\epsilon^{\frac{1}{2}}, \quad III \leq C\Delta_\epsilon^{\frac{1}{2}} - \frac{2}{\epsilon}\Delta_\epsilon \ .$$

Knowing that Δ_ϵ is dominated by ϵ^2, we are able to prove the Dirac form of $\bar{f}$. The convergence towards zero is not enough in itself, because to prove the special form $\bar{\rho}(x)\delta(v - F[\bar{\rho}, \bar{\jmath}])$ we need some information on a functional like

$$\bar{\Delta}_\epsilon(t) \ = \ \int_{\mathbb{R}^{2\mathrm{d}}} |v - F[\bar{\rho}, \bar{\jmath}]|^2 f_\epsilon(t, x, v) dx dv \ . \tag{19}$$

This new functional converges towards zero on all $[t^\star, \ T]$ whenever

$$\int_{\mathbb{R}^{2\mathrm{d}}} |F[\rho_\epsilon, j_\epsilon] - F[\bar{\rho}, \bar{\jmath}]|^2 f_\epsilon \ \longrightarrow \ 0 \quad in \ L^\infty([t^\star, \ T]),$$

and this is proved by the following lemma (notice that its hypothesis holds under the assumption of part (iii) of theorem 1)

Lemma 4 : *If Δ_ϵ is less than $C\epsilon^2$ on $[t_1, \ t_2]$, then $F[\rho_\epsilon, j_\epsilon]$ converges strongly towards $F[\bar{\rho}, \bar{\jmath}]$ in $C_0([t_1, \ t_2] \times \mathbb{R}^\mathrm{d})$. In particular if K, A and G belong to $W^{1,\infty}$ and if condition (9) is true, $F[\rho_\epsilon, j_\epsilon]$ converges strongly towards $F[\bar{\rho}, \bar{\jmath}]$ in $C_0([t^\star, \ T] \times \mathbb{R}^\mathrm{d})$ for all $t^\star > 0$.*

Proof of lemma 4

The only difficulty is again time continuity, which is dealt using the specific form of $F[\rho_\epsilon, j_\epsilon]$.

To prove the lemma, we first claim that

$$\partial_t j_\epsilon \ \in \ L^\infty([t_1, \ t_2], \ W^{-1,1}(R^d) + L^1(\mathbb{R}^\mathrm{d})).$$

This is deduced from equation (17) on j_ϵ and from the following inequality

$$\begin{aligned} \|j_\epsilon - \rho_\epsilon F[\rho_\epsilon, j_\epsilon]\|_{L^\infty([t_1, \ t_2], \ L^1(\mathbb{R}^\mathrm{d}))} \ &\leq \ \sup_{t \in [t_1, \ t_2]} \int_{\mathbb{R}^{2\mathrm{d}}} |v - F[\rho_\epsilon, j_\epsilon]| f_\epsilon(t, x, v) dx dv \\ &\leq \ (\sup \Delta_\epsilon(t))^{\frac{1}{2}} \ \|f^0\|_{L^1}^{\frac{1}{2}} \\ &\leq \ \tilde{C}\epsilon \ . \end{aligned}$$

Secondly, we use the bounds on $\partial_t j_\epsilon$ and $\partial_t \rho_\epsilon$ to conclude by Ascoli's theorem and arguments similar to those of lemma 1.

□

To complete the convergence proof of f_ϵ, we use lemma 5

Lemma 5 : *If $\bar\Delta_\epsilon(t)$ converges towards zero in $L^\infty([t^\star,\ T])$ for all $t^\star > 0$, then f_ϵ converges towards $\bar\rho\delta(v - F[\bar\rho, \bar\jmath])$ in $w - L^\infty([0,\ T], M^1(\mathrm{I\!R}^{2\mathrm{d}}))$.*

Proof of lemma 5

For ϕ and ψ in $C_0([t^\star,\ T] \times \mathrm{I\!R}^{\mathrm{d}})$, we have

$$\begin{aligned}\int_{\mathrm{I\!R}^{2\mathrm{d}}} \phi(t,x)\psi(t,v) f_\epsilon dx dv - \int_{\mathrm{I\!R}^{\mathrm{d}}} \phi\psi(t, F[\bar\rho,\bar\jmath])\rho_\epsilon dx \ &= \int_{|v-\bar F|<\eta} \phi(\psi(t,v) - \psi(t,\bar F)) f_\epsilon dx dv \\ &+ \int_{|v-\bar F|>\eta} \phi(\psi(t,v) - \psi(t,\bar F)) f_\epsilon dx dv \\ &\le\ C \sup_{t,|x-y|<\eta} |\psi(t,x) - \psi(t,y)| + \frac{C}{\eta^2}\int_{\mathrm{I\!R}^6} |v - F[\bar\rho,\bar\jmath]|^2 f_\epsilon dx dv \\ &\le\ C \sup_{|x-y|<\eta} |\psi(t,x) - \psi(t,y)| + \frac{C}{\eta^2}\sup_t \bar\Delta_\epsilon(t)\ .\end{aligned}$$

Since we already know that f_ϵ converges towards $\bar f$ in $w - L^\infty([0,\ T], M^1(\mathrm{I\!R}^{2\mathrm{d}}))$, ρ_ϵ towards $\bar\rho$ in $w - L^\infty([0,\ T], M^1(\mathrm{I\!R}^{\mathrm{d}}))$ and since $\psi(v - F[\bar\rho, \bar\jmath])$ is in $C_0([t^\star,\ T] \times \mathrm{I\!R}^{\mathrm{d}})$ (because ψ and $F[\bar\rho, \bar\jmath]$ belong to this space), the previous computation means that

$$f_\epsilon \ \rightharpoonup\ \bar\rho(t,x)\delta(v - F[\bar\rho,\bar\jmath]) \quad in\ L^\infty([t^\star,\ T], M^1(\mathrm{I\!R}^{2\mathrm{d}})),\ \forall \mathrm{t}^\star > 0.$$

Taking now any Φ in $L^1([0,\ T],\ C_0(\mathrm{I\!R}^{2\mathrm{d}}))$, we have

$$\int_0^T \int_{\mathrm{I\!R}^{2\mathrm{d}}} \Phi(t,x,v) f_\epsilon dx dv dt \ = \ \int_{t^\star}^T \int \Phi(t,x,v) f_\epsilon dx dv dt + \int_0^{t^\star} \int \Phi(t,x,v) f_\epsilon dx dv dt\ ,$$

and

$$\int_0^{t^\star} \int_{\mathrm{I\!R}^{2\mathrm{d}}} \Phi(t,x,v) f_\epsilon dx dv dt \le C \int_0^{t^\star} \|\Phi(t,.,.)\|_{C_0} dt \ \to\ 0 \ as\ t^\star \to 0,$$

which ends the proof.

□

2.4 Uniform bound for the kinetic energy

Here, we prove bounds for the kinetic energy under condition (9). we multiply (1) by $|v|^2$ and integrate in space and velocity, we find

$$\partial_t E_\epsilon(t) - \frac{2}{\epsilon}\int_{\mathrm{I\!R}^{\mathrm{d}}} j_\epsilon \cdot F[\rho_\epsilon, j_\epsilon] dx + \frac{2}{\epsilon} E_\epsilon(t) \ = \ 0\ .$$

Since

$$\int_{\mathbb{R}^d} j_\epsilon \cdot F[\rho_\epsilon, j_\epsilon] dx \;=\; \int_{\mathbb{R}^d} j_\epsilon \cdot (K \star j_\epsilon) dx + \int_{\mathbb{R}^d} j_\epsilon \cdot (A \star \rho_\epsilon) dx + \int_{\mathbb{R}^d} j_\epsilon \cdot G(x) dx \;,$$

the condition (9) gives

$$\partial_t E_\epsilon(t) \;\leq\; -\frac{2}{\epsilon} E_\epsilon(t) + \frac{2}{\epsilon} \int_{\mathbb{R}^d} j_\epsilon \cdot (A \star \rho_\epsilon) dx + \frac{2}{\epsilon} \int_{\mathbb{R}^d} j_\epsilon \cdot G(x) dx$$

The assumption (8) and the uniform bound of ρ_ϵ in $L^\infty([0,\ T],\ L^1(\mathbb{R}^d))$ imply that $A \star \rho_\epsilon$ and $G(x)$ are bounded in $L^\infty([0,\ T] \times \mathbb{R}^d)$ and thus

$$\partial_t E_\epsilon(t) \;\leq\; -\frac{2}{\epsilon} E_\epsilon(t) + \frac{C}{\epsilon} \|j_\epsilon\|_{L^1(\mathbb{R}^d)} \;,$$

using (16), we deduce from this last inequality that

$$E_\epsilon(t) \;\leq\; \mathrm{Min}\left(E^0, \frac{C^2}{4}\right) \;.$$

Remark.

The condition (9) is used to deal with the quadratic term in j_ϵ. However, just like in lemma 3, here we can replace this condition by the smallness assumption $\|K\|_{L^\infty} \|f^0\|_{L^1} < 1$ and still get the boundedness of kinetic energy.

3. Proof of theorem 2

We first deal with the existence problem and in a second part we will prove the uniqueness result.

3.1 Existence of solutions in L^1

First of all notice that with the condition (9) the second equation of system (5) has only one solution j in L^1 for a given ρ. Indeed for two solutions j_1 and j_2, the difference $j = j_1 - j_2$ satisfies, thanks to (7), the equation

$$j(t,x) \;=\; \rho(t,x)(K \star j) \;.$$

Multipliing the equation by j/ρ (which exists and belongs to C_0 since it is equal to $K \star j$) and integrating, we find

$$\int_{\mathbb{R}^d} \frac{|j|^2}{\rho} dx \;=\; \int_{\mathbb{R}^d} j \cdot (K \star j) dx \;\leq\; 0 \;,$$

which means that $j = 0$.

Theorem 1 provides an existence result for an initial data ρ^0 in $C_0(\mathbb{R}^d)$. Indeed in that case it is very easy to find a function f^0 in $L^1 \cap L^\infty$, with bounded kinetic energy, which two first velocity moments are ρ_0 and j^0. For example we can take for f^0 a local maxwellian in velocity. Since we satisfy the asssumptions of theorem 1, we get a couple ρ, j in $L^\infty([0,\ \infty[\,, L^1(\mathrm{I\!R}^d))$, solution to (5). To complete the existence proof in L^1, we only need a stability result for the system (5) which is given by lemma 6.

Lemma 6 : *Let ρ_n, $j_n \in L^\infty([0,\ \infty[\,, L^1 \cap L^p(\mathrm{I\!R}^d))$, $p > 1$, be two sequences of distributional solutions to* (5). *Assume* (9), *and that $\rho_n^0 \to \rho^0$ in $L^1(\mathrm{I\!R}^d)$. Then, extracting subsequences, ρ_n and j_n converge weakly in $L^\infty([0,\ \infty[\,, M^1(\mathrm{I\!R}^d))$ to ρ and j. Also, ρ and j belong to $L^\infty([0,\ \infty[\,, L^1(\mathrm{I\!R}^d))$ and are solution to the system* (5).

Proof of lemma 6

Since ρ_n^0 converges strongly in L^1, ρ_n is bounded in $L^\infty([0,\ \infty[\,, L^1(\mathrm{I\!R}^d))$. Multipliing the second equation of (5) by j_n/ρ_n and integrating, we find thanks to (9)

$$\int_{\mathrm{I\!R}^d} \frac{|j_n|^2}{\rho_n} dx \ \leq \ \int_{\mathrm{I\!R}^d} j \cdot (A \star \rho_n + G(x)) dx \ ,$$

which gives a uniform bound on $|j_n|^2/\rho_n$ in $L^\infty([0,\ \infty[\,, L^1(\mathrm{I\!R}^d))$. We deduce a bound on j_n in the same space

$$\int_{\mathrm{I\!R}^d} |j_n(t,x)| dx \ \leq \ \left(\int_{\mathrm{I\!R}^d} \frac{|j_n|^2}{\rho_n} dx \right)^{\frac{1}{2}} \cdot \left(\int_{\mathrm{I\!R}^d} \rho_n dx \right)^{\frac{1}{2}} \ .$$

Extracting subsequences if necessary, we now suppose that ρ_n and j_n converge towards ρ and j in $L^\infty([0,\ \infty[\,, M^1(\mathrm{I\!R}^d))$. The first equation of system (5) is linear in ρ_n and j_n, so we can pass to the limit and we find in distributional sense

$$\partial_t \rho + \mathrm{div} j \ = \ 0 \ .$$

We now consider the limit of the term $\rho_n F[\rho_n, j_n]$. The only difficulties arise in $\rho_n (K \star j_n)$ and $\rho_n (A \star \rho_n)$. We use the lemma 1 for ρ_n and j_n instead of ρ_ϵ and j_ϵ and as a consequence $\rho_n F[\rho_n, j_n]$ converges towards $\rho F[\rho, j]$ in $w^\star - L^\infty([0,\ \infty[,\ M^1(\mathrm{I\!R}^d))$ and we get the second equation of the system (5)

$$j(t,x) \ = \ \rho(t,x) F[\rho, j] \ .$$

To prove that ρ and j are functions and not only measures, we notice that, since ρ^0 is in L^1, there exists a function β from $\mathrm{I\!R}_+$ to $\mathrm{I\!R}_+$ with

$$\beta(x) \ \geq \ |x|, \quad \frac{\beta(x)}{|x|} \ \longrightarrow \ \infty \ as \ |x| \to \infty \ ,$$

and such that

$$\int_{\mathrm{I\!R}^d} \beta(\rho^0(x)) dx \ < \ \infty \ .$$

Regularizing ρ_n if necessary, we can suppose that $\int_{\mathbb{R}^3} \beta(\rho_n^0)$ are uniformly bounded. Then since ρ_n are solutions to the system (5) and since the divergences of K, A and G are bounded, the quantities $\int_{\mathbb{R}^d} \beta(\rho_n)$ are uniformly bounded in $L^\infty\,[0,\ \infty[$ and finally

$$\int_{\mathbb{R}^d} \beta(\rho(t,x))dx \ \in\ L^\infty\,[0,\ \infty[\ ,$$

which shows that ρ belongs to $L^\infty([0,\ \infty[\,, L^1(\mathbb{R}^d))$. Using the second equation of the system (5), we find that j has the same property. Notice that this proves that we have weak convergence in $L^\infty([0,\ T],\ L^1)$.

□

3.2 Uniqueness in $W^{1,1}(\mathbb{R}^d)$

Choose any ρ^0 in $W^{1,1}(\mathbb{R}^d)$ with

$$\|\rho^0\|_{L^1(\mathbb{R}^d)} \times \|K\|_{L^\infty(\mathbb{R}^d)} \ <\ 1, \quad \|\rho^0\|_{L^1(\mathbb{R}^d)} \times \|\text{div } K\|_{L^\infty(\mathbb{R}^d)} \ <\ 1\ .$$

Suppose that we have two couples of solutions (ρ_1, j_1) and (ρ_2, j_2) to the system (5) with initial data ρ^0 in $L^\infty([0,\ \infty[\ ,\ L^1(\mathbb{R}^d))$. Applying lemma 7 shows that these quantities are also in $L^\infty([0,\ T],\ W^{1,1}(\mathbb{R}^d))$ for a time T depending only on the initial data and on the norms of K, A and G in L^∞. We prove uniqueness only on this time interval.

Lemma 7 : *For all constant C with $C\|K\|_{L^\infty} < 1$, there exists a time T such that if $\|\rho^0\|_{L^1} \le C$ and if ρ^0 belongs to $W^{1,1}$, then any solution ρ in $L^\infty([0,\ \infty[\ ,\ L^1(\mathbb{R}^d))$ to the system* (5) *belongs to $L^\infty([0,\ T],\ W^{1,1}(\mathbb{R}^d))$.*

Proof of lemma 7

We first differentiate the equation (7)

$$\nabla F[\rho,\ j] \ =\ A \star \nabla\rho + K \star \nabla j + \nabla G\ .$$

Using now the second equation of (5), we find

$$\nabla F[\rho,\ j] - K \star (\rho \nabla F[\rho, j]) \ =\ A \star \nabla\rho + \nabla G\ .$$

Thanks to the smallness assumption on the L^1 norm of ρ^0 and thus on the L^1 bound on ρ, we deduce the estimate

$$\|\nabla F[\rho,\ j]\|_{L^\infty} \ \le k\big(\|\nabla\rho(t,.)\|_{L^1(\mathbb{R}^d)} + 1\big)\ .$$

Now, we replace j by $\rho F[\rho, j]$ and we differentiate the continuity equation in (5)

$$\partial_t \nabla\rho + \text{div}(\rho\nabla F[\rho, j]) + \text{div}(\nabla\rho F[\rho, j]) \ =\ 0\ ,$$

which shows that

$$\frac{d}{dt}\int_{\mathbb{R}^d} |\nabla\rho| dx \;\leq\; \int_{\mathbb{R}^d} |\nabla F[\rho, j]| \cdot |\nabla\rho| dx \;\leq\; k\|\nabla\rho(t,.)\|_{L^1(\mathbb{R}^d)} \;,$$

the constant k depending only on the bound on $\|\rho^0\|_{L^1}$ and on the norms of K, A and G.

As a consequence there exists a constant T depending only on these quantities such that ρ belongs to $L^\infty([0,\ T],\ W^{1,1}(\mathbb{R}^d))$.

□

Notice that of course if K and A are in $W^{1,\infty}$, the same proof holds without any assumption on the L^1 norm of ρ^0 and for all times.

We are now able to prove uniqueness. Let us substract $F[\rho_1, j_1]$ and $F[\rho_2, j_2]$ by using formula (7)

$$F[\rho_1, j_1] - F[\rho_2, j_2] \;=\; A \star (\rho_1 - \rho_2) + K \star (j_1 - j_2) \;,$$

and denoting $F[\rho_1, j_1]$ by F_1 and $F[\rho_2, j_2]$ by F_2, since

$$K \star (j_1 - j_2) \;=\; K \star (\rho_1 F_1 - \rho_2 F_2) \;=\; K \star (\rho_1(F_1 - F_2)) + K \star ((\rho_1 - \rho_2)F_2) \;,$$

we have

$$F_1 - F_2 - K \star (\rho_1(F_1 - F_2)) \;=\; A \star (\rho_1 - \rho_2) + K \star ((\rho_1 - \rho_2)F_2) \;.$$

The bound $\|\rho^0\|_{L^1}\|K\|_{L^\infty} < 1$ gives the estimate

$$\|(F_1 - F_2)(t,.)\|_{L^\infty(\mathbb{R}^d)} \;\leq\; k\|(\rho_1 - \rho_2)\|_{L^1(\mathbb{R}^d)} \;.$$

Moreover taking the divergence of the previous expression and since we also have $\|\rho^0\|_{L^1}\|\mathrm{div}\ K\|_{L^\infty} < 1$, we obtain

$$\|\mathrm{div}(F_1 - F_2)(t,.)\|_{L^\infty(\mathbb{R}^d)} \;\leq\; k\|(\rho_1 - \rho_2)\|_{L^1(\mathbb{R}^d)} \;.$$

Eventually, we substract the two continuity equations satisfied by ρ_1 and ρ_2 with j_1 and j_2 replaced by F_1 and F_2 and we get

$$\partial_t(\rho_1 - \rho_2) + \mathrm{div}\big((\rho_1 - \rho_2)F_1\big) + \mathrm{div}\big(\rho_2(F_1 - F_2)\big) \;=\; 0 \;,$$

which leads to

$$\frac{d}{dt}\int_{\mathbb{R}^d} |\rho_1 - \rho_2| dx \;\leq \int_{\mathbb{R}^d} |F_1 - F_2| \cdot |\nabla\rho_2| dx + \int_{\mathbb{R}^d} |\mathrm{div}(F_1 - F_2)| \cdot |\rho_2| dx \;,$$

and combining this with the previous estimates

$$\frac{d}{dt}\|(\rho_1 - \rho_2)(t,.)\|_{L^1(\mathbb{R}^d)} \;\leq\; k\|(\rho_1 - \rho_2)(t,.)\|_{L^1(\mathbb{R}^d)} \;.$$

Gronwall lemma implies

$$\rho_1 \;=\; \rho_2 \ ,$$

and the proof of theorem 2 is complete.

4. Proof of theorem 3

Throughout this section, we consider the system (6) with the matrix K given by formula (4) (in particular we work in dimension 3). The first part of the proof is devoted to getting j as a function of ρ with the second equation of (6), after that we prove the existence result and the uniqueness. These last two subsections use the same methods as in section 3 and so we do not write the details.

4.1 The second equation of (6)

Here we show the following lemma

Lemma 8 : *Assume that $\rho(x)$ belongs to $L^1 \cap L^{\frac{3}{2}}(\mathbb{R}^3)$. Then, there is a function $j(x)$ in $L^1 \cap L^{\frac{3}{2}}$ such that the second equation of* (6) *is satisfied. Moreover this function is unique in every L^p, $1 \le p \le \frac{3}{2}$.*

Proof of the lemma

First of all notice that the singularity in K is in $1/|x|$. As a consequence j lies in L^p with $p \le \frac{3}{2}$ to define the term $K \star j$.

Now if ρ belongs to $L^1 \cap L^{\frac{3}{2}}$, the product $\rho(K \star j)$ is well defined for any j in L^p with $1 \le p \le \frac{3}{2}$. Indeed $K \star j$ belongs to L^q with $1/q = 1/p - 2/3$ and so q is always between 3 and ∞.

To prove the existence of a j in $L^1 \cap L^{\frac{3}{2}}$, we use an iterative procedure. We define the following sequence

$$\begin{aligned} j_0(x) \;&=\; g\rho(x) \ , \\ j_{n+1}(x) \;&=\; \rho\big(K \star j_n + g\big) \ . \end{aligned}$$

We assume that ρ is small enough in $L^1 \cap L^{\frac{3}{2}}$. More precisely for $|K| \le c/|x|$, we assume

$$C = c\|\rho\|_{L^1 \cap L^{\frac{3}{2}}} < 1 \ .$$

Since we have

$$\|j_{n+1}\|_{L^1 \cap L^{\frac{3}{2}}} \;\le\; C\big(\frac{|g|}{c} + \|j_n\|_{L^1 \cap L^{\frac{3}{2}}}\big) \ ,$$

we deduce that the sequence j_n is uniformly bounded in $L^1 \cap L^{\frac{3}{2}}$. Hence we extract a subsequence weakly converging in $L^1 \cap L^{\frac{3}{2}}$ towards a function j. The convolution $K \star j_n$ thus converges strongly towards $K \star j$ in $L^3 \cap L^\infty$ and so we obtain

$$j(x) \;=\; \rho(x)\big(K\star j+g\big)\ .$$

Eventually if ρ is not small, then we use this argument to find j such that

$$j(x) \;=\; N^2\rho(Nx)\big(K\star j+\frac{g}{N^2}\big)\ ,$$

with N large enough so that $\|N^2\rho(Nx)\|_{L^1}$ is small enough (we work in dimension 3) and the function $j(x/N)$ satisfies the second equation of (6) with ρ.

The uniqueness of such a function is proved exactly as in the beginning of subsection 3.1 because the matrix K given by (4) satisfies condition (9).

$\square$

4.2 Existence of solutions

Let us denote K_δ a sequence of regularisations of K in $W^{1,\infty}$. Theorem 2 provides the existence of ρ_δ and j_δ, solutions to (5) with K_δ, $A=0$ and $G=g$. As a consequence, we prove that these two sequences converge towards the solution of (6).

Since j_δ/ρ_δ is divergence free, the continuity equation implies that

$$\|\rho_\delta\|_{L^\infty([0,\ \infty],L^1\cap L^{\frac32})} \;\leq\; \|\rho^0\|_{L^1\cap L^{\frac32}}\ .$$

Now using lemma 8, j_δ is uniformly bounded in $L^\infty([0,\ \infty],\ L^1\cap L^{\frac32})$. We then extract subsequences (still denoted ρ_δ and j_δ) which converge weakly in $L^\infty([0,\ \infty],\ L^1\cap L^{\frac32})$ towards ρ and j.

The couple ρ_δ, j_δ satisfying the continuity equation, it is also true for ρ, j. As to the second equation of (6), we have to prove that the term $\rho_\delta(K_\delta\star j_\delta)$ converges weakly towards $\rho(K\star j)$. This is done as in lemma 1, since the convolution provides compactness in space and the continuity equation provides compactness in time.

4.3 Uniqueness

We show first that the system (6) propagates the $W^{1,3}$ norm of ρ in small time.

Lemma 9 : *Assume that* $\rho^0\in W^{1,1}\cap W1,3$. *Then, there exists a time* T *such that for any solution* ρ *to* (6) *obtained by weak limit of classical solution to a regularisation of* (6), *we have* $\rho\in L^\infty([0,\ T],W^{1,1}\cap W^{1,3}(\mathbb{R}^3))$.

Proof of the lemma

We use the same ideas as in lemma 7. By Sobolev inequalities, for all $p<\infty$, $\|\rho\|_{L^p_x}$ is less than $\|\rho\|_{L^1\cap W^{1,3}}$. From the second equation of (6) and the bounds on $\|j\|_{L^{\frac32}}$ and $\|\rho\|_{L^1}$, we deduce the a priori estimate for all $p<3$

$$\|j\|_{W^{1,p}} \;\leq\; C\|\rho\|_{W^{1,1}\cap W^{1,3}}\ .$$

This new estimate implies that

$$\|K\star j\|_{W^{1,\infty}} \ \leq \ C\|\rho\|_{W^{1,1}\cap W^{1,3}} \ .$$

After differentiating the continuity equation, we eventually find for all $1\leq p\leq 3$

$$\frac{d}{dt}\|\rho(t,.)\|_{W^{1,p}} \ \leq \ C\|\rho(t,.)\|_{W^{1,p}}\times\|\rho(t,.)\|_{W^{1,1}\cap W^{1,3}} \ ,$$

which by Gronwall lemma means that $\|\rho\|_{W^{1,1}\cap W^{1,3}}$ remains bounded on an interval $[0,\ T]$, T depending on $\|\rho^0\|_{W^{1,1}\cap W^{1,3}}$. Of course this computation is only formal. However we assume that ρ is a weak limit of classical solution to (6) with a regularized K and thus we can make it rigorous.

□

Consider now any ρ^0 in $L^1\cap W^{1,1}\cap W^{1,3}$. Assume we have two couples (ρ_1, j_1) and (ρ_2, j_2) in $L^1\cap L^{\frac{3}{2}}$ of solutions to (6) with initial data ρ^0. We suppose that ρ_1 satisfies the assumption of lemma 9 (this is possible because section 4.2 ensures that such a solution exists). Hence ρ_1 belongs to $L^\infty([0,\ T],\ W^{1,1}\cap W^{1,3})$ for some time T and we will prove that $\rho_1=\rho_2$ on this time interval.

We first estimate j_1-j_2 in term of $\rho_1-\rho_2$

$$j_1-j_2 \ = \ (\rho_1-\rho_2)(K\star j_1+g)+\rho_2(K\star(j_1-j_2)) \ ,$$

using now the uniform $L^1\cap L^{\frac{3}{2}}$ on ρ_i and j_i, $i=1,2$, a minor modification of lemma 8 gives

$$\|j_1-j_2\|_{L^\infty([0,\ T],\ L^{\frac{3}{2}})} \ \leq \ C\|\rho_1-\rho_2\|_{L^{\frac{3}{2}}} \ .$$

We replace the j_i in the two continuity equations by their value given by the second equation of (6) and we substract using the divergence free condition of K

$$\partial_t(\rho_2-\rho_1)+(K\star j_2+g)\cdot\nabla(\rho_2-\rho_1)+(K\star(j_2-j_1))\cdot\nabla\rho_2 \ = \ 0 \ ,$$

multiplying by $(\rho_2-\rho_1)/|\rho_2-\rho_1|^{\frac{1}{2}}$, integrating and using Holder inequalities and the previous estimate for j_1-j_2, we find

$$\partial_t\|(\rho_2-\rho_1)(t,.)\|_{L^{\frac{3}{2}}} \ \leq C\|(\rho_2-\rho_1)(t,.)\|_{L^{\frac{3}{2}}} \ ,$$

with C a constant depending on the $W^{1,1}\cap W^{1,3}$ norm of ρ_2 and on the $L^1\cap L^{\frac{3}{2}}$ norms of ρ_1, ρ_2, j_1, j_2. To end the proof of theorem 3, we apply Gronwall lemma to show that $\rho_1 \ = \ \rho_2$.

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