# Lemmes de moyenne et Transformée aux rayons X

# Averaging Lemmas and the X-ray transform

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**Abstract.** We introduce a new method to prove averaging lemmas, *i.e.* prove a regularizing effect on the average in velocity of a solution to a kinetic equation. The method does not require the use of Fourier transform and the whole procedure is performed in the 'real space'; It leads to estimating an operator very similar to the so-called X-ray transform. We are then able to improve the known results when the integrability in space and velocity is different.

**Résumé.** Nous introduisons une nouvelle méthode pour obtenir des lemmes de moyennes, c'est-à-dire prouver un effet régularisant pour les moyennes en vitesse d'une équation cinétique. Cette méthode ne fait pas appel à la transformée de Fourier et toute la preuve se fait dans "l'espace réel" ; on est alors

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conduit à borner un opérateur très semblable à la transformée aux rayons X. Nous améliorons grâce à cela les résultats déjà connus quand l'intégrabilité en espace et en vitesse est différente.

**Version française abrégée.**
On s'intéresse à l'équation cinétique stationnaire suivante

$$v \cdot \nabla_x f(x,v) = \Delta^{\alpha/2}\, g(x,v), \qquad x \in \mathbb{R}^d,\ v \in \mathbb{R}^d,\ 0 \le \alpha < 1,$$

et plus précisemment à la régularité des moyennes en vitesse de la solution. Ainsi on pose

$$\rho(x) = \int_{\mathbb{R}^d} f(x,v)\phi(v)\,dv, \qquad \phi \in C_c^\infty(\mathbb{R}^d) \text{ donnée.}$$

On sait depuis l'article de Golse, Perthame et Sentis [7] que l'on observe un effet régularisant de l'équation sur la moyenne $\rho$. Le résultat optimal que l'on peut prouver pour $f$ et $g$ dans $L^p(\mathbb{R}^{2d})$ a été obtenu par DiPerna, Lions et Meyer dans [5] (voir aussi [3]). Cependant l'article de Westdickenberg [14] suggère que l'on peut améliorer le résultat pour peu que $f$ ou $g$ appartiennent à un espace du type $L^p_x(L^q_v)$ ou le contraire et cela fait l'objet de l'étude de [10]. Ainsi supposons que

$$f \;\in\; W_v^{\beta,\,p_1}(\mathbb{R}^d,\ L_x^{p_2}(\mathbb{R}^d)),\ \beta \ge 0, \quad g \;\in\; W_v^{\gamma,\,q_1}(\mathbb{R}^d,\ L^{q_2}(\mathbb{R}^d)),\ -\infty < \gamma < 1.$$

On prouve dans [10] le théorème suivant

**Théorème** *Si* $1 < p_2, q_2 < \infty$, $1 \le p_1 \le \min(p_2,\ p_2^*)$ *et* $1 \le q_1 \le \min(q_2,\ q_2^*)$ *où* $p^*$ *est l'exposant dual de* $p$*, et si de plus* $\gamma - 1/q_1 < 0$*, alors pour tout* $s' < s$

$$\rho \in \dot W^{s',r}, \quad \text{avec } \frac{1}{r} = \frac{1-\theta}{p_2} + \frac{\theta}{q_2}, \quad s = (1-\alpha) \times \frac{1+\beta-1/p_1}{1+\beta-1/p_1-\gamma+1/q_1}.$$

Quand l'ordre d'intégration en $x$ et $v$ est différent, la situation est plus complexe mais on peut par exemple prouver le résultat suivant

**Proposition** *En dimension deux, si* $g(,.v)\phi(v)$ *est paire en* $v$ *et si* $f,\ g \in L_x^{4/3}(\mathbb{R}^2,\ L_v^2(\mathbb{R}^2))$ *alors* $\rho \in W^{s,4/3}(\mathbb{R}^2)$ *pour tout* $s < 1/2$.
*Si* $f,\ g \in L_x^{4/3}(\mathbb{R}^2, L_v^\infty(\mathbb{R}^2))$*, alors on a même* $\rho \in H^s(\mathbb{R}^2)$ *pour tout* $s < 1/2$.

L'originalité de la méthode repose sur le fait que tout se passe dans l'espace réel sans utiliser l'analyse de Fourier. En fait si $g$ est à support compact, il est aisé de voir que l'expression suivante

$$U\,g = \int_{v\in\mathbb{R}^d}\int_0^{+\infty} g(x-vt,v)\,dt\,\phi(v)\,dv$$

est la moyenne d'une solution de l'équation cinétique. Or l'opérateur $U$ est à peu de chose près le dual de la transformée aux rayons X qui s'écrit

$$(T\,h)(x,v) = \int_{-\infty}^{+\infty} h(x-vt)\,dt.$$

Il suffit alors d'avoir des estimations pour des opérateurs comme $T$ pour obtenir des lemmes de moyenne.

# 1 Introduction

All the details concerning the results and the proofs can be found in [10]. This paper is concerned with the following stationary kinetic equation

$$v\cdot\nabla_x f(x,v) = \Delta^{\alpha/2}\,g(x,v), \qquad x\in\mathbb{R}^d,\ v\in\mathbb{R}^d,\, 0\le\alpha<1, \tag{1.1}$$

and particularly with the regularity of the velocity averages of the solution,

$$\rho(x) = \int_{\mathbb{R}^d} f(x,v)\phi(v)dv \text{ or } \quad \rho = \int_{S^{d-1}} f(x,v)\phi(v)dv,\ \phi\in C_c^\infty \text{ given.} \tag{1.2}$$

Assume that

$$f\in W_v^{\beta,p_1}(\mathbb{R}^d,\ L_x^{p_2}(\mathbb{R}^d)),\ \beta\ge 0, \quad g\in W_v^{\gamma,q_1}(\mathbb{R}^d,\ L^{q_2}(\mathbb{R}^d)),\ \gamma<1. \tag{1.3}$$

Then we can prove the theorem

**Theorem 1.1** *Let $f$ and $g$ satisfy (1.1) and (1.3) with $1<p_2,\,q_2<\infty$, $1\le p_1\le\min(p_2,\ p_2^*)$ and $1\le q_1\le\min(q_2,\ q_2^*)$ where $p^*$ is the dual exponent of $p$, and assume moreover that $\gamma-1/q_1<0$. Then, for any $s'<s$*

$$\rho\in\dot W^{s',r}, \quad \textit{with } \frac{1}{r} = \frac{1-\theta}{p_2}+\frac{\theta}{q_2},\ s=(1-\alpha)\times\frac{1+\beta-1/p_1}{1+\beta-1/p_1-\gamma+1/q_1}.$$

When the order between the spaces in (1.3) is interchanged, the situation becomes more complex. However a typical result is in that case

**Proposition 1.1** *In dimension two, if $g(.,v)\phi(v)$ is even in $v$ and if $f,\ g \in L_x^{4/3}(\mathbb{R}^2,\ L_v^2(\mathbb{R}^2))$ satisfy (1.1), then $\rho \in W^{s,4/3}(\mathbb{R}^2)$, for any $s < 1/2$.*
*If $f,\ g \in L_x^{4/3}(\mathbb{R}^2,\ L_v^\infty(\mathbb{R}^2))$ satisfy (1.1), then $\rho \in H^s(\mathbb{R}^2)$, for any $s < 1/2$.*

Note that in any case, it is indifferent whether the average is taken in $\mathbb{R}^d$ or only on the sphere.
Averaging lemmas were first proved in a $L^2$ framework in [7] by Golse, Perthame and Sentis. They were later extended to get an optimal result for $f$ and $g$ in $L^p_{x,v}$ by DiPerna, Lions and Meyer in [5] (see also Bouchut [3] for a simpler proof). They are important because in many applications the important and physical quantity is some average of the solution. They were for instance used to get weak solutions to the Vlasov-Maxwell system in [4]. But they maybe find their main application where kinetic formulations appear, whether for scalar conservation laws, isentropic gas dynamics (see [11] and [12] by Lions, Perthame and Tadmor or the book by Perthame [13]) or line-energy Ginzburg-Landau models (see [8]).
Although they are optimal in their classical version, averaging lemmas can be improved when more information is known on $f$ or $g$. For instance if one of these functions belongs to a Sobolev space in velocity, then it was noticed in [9] (and later for the solution $f$ itself in [2]) that the average is more regular. The study of [10] was motivated by a paper of Westdickenberg [14] which suggests that the number of derivatives gained on the average depends only on the regularity (derivability or integrability) in velocity of $f$ or $g$.

# 2 The connection with the X-ray transform

From Eq. (1.1), we obviously have for any $\lambda > 0$

$$(\lambda + v \cdot \nabla_x)\, f = g + \lambda f.$$

But now the operator $\lambda + v \cdot \nabla_x$ is invertible and we may write

$$\rho = Sg + \lambda Sf, \text{ with } \quad Sf(x) = \int_{\mathbb{R}^2} \int_0^\infty f(x - vt, v)\, e^{-\lambda t}\, \Phi(v)\, dt\, dv.$$

If we are able to estimate $Sg$ and $Sf$, then by the method of real interpolation we can estimate the regularity of the average (see the book by Bergh and Löfström [1] or [9] where it was used for the first time for averaging lemmas). Thus the essential part is the study of $S$, or of its dual operator which reads

$$(S^*h)(x,v) = \phi(v) \times \int_0^\infty h(x+vt)\, e^{-\lambda t}\, dt.$$

This operator is very similar to the X-ray transform which is defined by

$$(T\,h)(x,v) = \int_{-\infty}^{+\infty} h(x-vt)\, dt. \tag{2.1}$$

Studies of $T$ have already been done but they were more concerned with the integrability properties than with gain of derivatives (see the papers by Wolff [15] and by Duoandikoetxea and Oruetxebarria [6] for instance).

# 3 Sketch of the proof of Prop. 1.1

We in fact prove the following estimate on $T$

**Lemma 3.1** *For any set $E \subset B(0,K)$ and any $0 \le s < 1/2$*

$$\|\Delta_x^{s/2} T \mathbb{I}_E\|^4_{L^4_x(B(0,K),\, L^2_v(S^1))} \le C(K)|E|.$$

This lemma implies that $T$ is continuous from $L^4_{loc}$ to $W^{s/2,4}_{x,loc}(L^2_v)$ (first we get the estimates with norms of Lorentz spaces for any function and by Sobolev embedding ($s < 1/2$) the result). This is the core of the proof to show that $S^*$ shares the same property which eventually gives Proposition 1.1, and it is where the evenness condition is needed.

**Proof of Lemma 3.1.** We decompose the sphere $S^1$ into subdomains $S_k$ with $k = 1, 2$ such that $|v_k| > 1/2$ in $S_k$. Of course it is enough to prove Lemma 3.1 with $S_k$ instead of $S^1$ and by symmetry we do it only for $S_1$.
Then we write

$$\begin{aligned}
\|\Delta_x^{s/2} T \mathbb{I}_E\|^4_{L^4_x(B(0,K),\, L^2_v(S_1))} &= \int_{B(0,K)} \left( \int_{v\in S_1} |\Delta_x^{s/2} T \mathbb{I}_E(x,v)|^2\, dv \right)^2 dx \\
&= \int_{B(0,K)} \int_{v,w\in S_1} |\Delta_x^{s/2} T \mathbb{I}_E(x,v)|^2 \times |\Delta_x^{s/2} T \mathbb{I}_E(x,w)|^2\, dv\, dw\, dx \\
&= \int_{v\in S_1} \int_{x\in B(0,K)} \int_{w\in S_1} |\Delta_x^{s/2} T \mathbb{I}_E(x,v)|^2\, |\Delta_x^{s/2} T \mathbb{I}_E(x,w)|^2\, dw\, dx\, dv.
\end{aligned}$$

We change variables in $x$ decomposing $x$ in $y + lv$ with $y$ in the plane $H_1$ of equation $x_1 = 0$. Since $|v_1| > 1/2$, the jacobian of the transformation is bounded and as all the terms in the integral are non negative, we have

$$\|\Delta_x^{s/2} T\mathbb{I}_E\|^4_{L^4_x(B(0,K)),\,L^2_v(S^1))} \leq \int_{v\in S_1}\int_{y\in H_1}\int_{l=-K}^{K}\int_{w\in S_1} |\Delta_x^{s/2} T\,\mathbb{I}_E(y+lv, v)|^2 \times |\Delta_x^{s/2} T\mathbb{I}_E(y+lv, w)|^2\, dw\, dl\, dy\, dv$$

$$\leq \int_{v\in S_1}\int_{y\in H_1} |\Delta_x^{s/2} T\,\mathbb{I}_E(y,v)|^2 \left(\int_{l=-K}^{K}\int_{w\in S_1} |\Delta_x^{s/2} T\mathbb{I}_E(y+lv, w)|^2 dw\, dl\right) dy\, dv,$$

because $Tf(x,v)$ is constant on any line with direction $v$ and therefore $\Delta_x^{s/2} T\,\mathbb{I}_E(y+lv, v)$ does not depend on $l$. We denote

$$I(y,v) = \int_{l=-K}^{K}\int_{w\in S_1} |\Delta_x^{s/2} T\mathbb{I}_E(y+lv, w)|^2\, dw\, dl.$$

We will show that $I$ belongs to $L^\infty$. We fix $y$ and $v$ and we decompose $S_1$ into the union of $S_1^i$ with $S_1^i = \{w \in S^1,\ 2^{-i-1} < |v - w| < 2^{-i}\}$, so that

$$I(l,v) = \sum_{i=0}^{\infty} I_i(l,v) = \sum_{i=0}^{\infty} \int_{l=-K}^{K}\int_{w\in S_1^i} |\Delta_x^{s/2} T\,\mathbb{I}_E(y+lv, w)|^2\, dw\, dl.$$

Of course $\Delta_x^{s/2} T\,\mathbb{I}_E(y+lv, w)$ is constant along any line with direction $w$ so we may bound

$$I_i \leq \frac{1}{2K}\int_{w\in S_1^i}\int_{l=-K}^{K}\int_{s=-K}^{K} |\Delta_x^{s/2} T\,\mathbb{I}_E(y+sw+lv, w)|^2\, ds\, dl\, dw.$$

We again change variables from $l$ and $s$ to $z = y + sw + lv$. We denote by $C_{y,v,w}$ the set $\{y + sw + lv,\ |s| \leq K,\ |l| \leq K\}$ and by $|(v,w)|$ the sinus of the angle between $v$ and $w$. Then

$$\begin{aligned} I_i &\leq \frac{1}{2K}\int_{w\in S_1^i}\int_{z\in C_{y,v,w}} |\Delta_x^{s/2} T\,\mathbb{I}_E(z,w)|^2 \frac{dz\, dw}{|(v,w)|} \\ &\leq \frac{2^{i+1}}{2K}\int_{w\in S_1^i}\int_{z\in C_{y,v,w}} |\Delta_x^{s/2} T\,\mathbb{I}_E(z,w)|^2\, dz\, dw. \end{aligned}$$

Denote $C_{y,v} = \bigcup_{w\in S_1^i} C_{y,v,w}$ and $\tilde{E} = E \cap C_{y,v}$. Clearly, as all the terms are non negative

$$I_i \leq \frac{2^{i+1}}{2K}\int_{w\in S_1^i}\int_{z\in C_{y,v}} |\Delta_x^{s/2} T\,\mathbb{I}_{\tilde{E}}(z,w)|^2\, dz\, dw.$$

Using a Hölder estimate, we find for any $p > 2$,

$$
\begin{aligned}
I_i &\leq \frac{2^{i+1}}{2K} \times |C_{y,v}|^{1-2/p} \times \int_{w \in S_1^i} \left( \int_{z \in C_{y,v}} |\Delta_x^{s/2} T\, \mathbb{I}_{\tilde{E}}(z,w)|^p \, dz \right)^{2/p} dw \\
&\leq C(K)\, 2^{i+1} \times 2^{-i(1-2/p)} \times \int_{w \in S^1} \left( \int_{z \in B(0,2K)} |\Delta_x^{s/2} T\, \mathbb{I}_{\tilde{E}}(z,w)|^p \, dz \right)^{2/p} dw,
\end{aligned}
$$

because the measure of $C_{y,v}$ is bounded by $2^{-i}$ times a constant depending on $K$. By Sobolev embedding, for $1/2 - (1/2 - s)/2 \leq 1/p < 1/2$, the last integral is dominated by the $L^2_w H^{1/2}_z$ norm of $T\mathbb{I}_{\tilde{E}}$. Therefore, taking $1/p = 1/2 - (1/2 - s)/2$

$$
\begin{aligned}
I_i &\leq C(K)\, 2^{i+1} \times 2^{-i(1/2-s)} \times \int_{w \in S^1} \int_{z \in B(0,2K)} |\Delta_x^{1/4} T\, \mathbb{I}_{\tilde{E}}(z,w)|^2 \, dz\, dw \\
&\leq C(K)\, 2^{i+1} \times 2^{-i(1/2-s)} \times C\, |\tilde{E}| \leq C(K) \times 2^{-i(1/2-s)},
\end{aligned}
$$

because the measure of $\tilde{E}$ is less than the measure of $C_{y,v}$. Eventually we may sum up the series and get

$$
I = \sum_{i=0}^{\infty} I_i \leq C(K).
$$

Using again the known $L^2$ estimate on $T$, this has as immediate consequence

$$
\begin{aligned}
\|\Delta_x^{s/2} T \mathbb{I}_E\|^4_{L^4_x(B(0,K),\, L^2_v(S_1))} &\leq C(K) \int_{v \in S_1} \int_{y \in H_1} |\Delta_x^{s/2} T\, \mathbb{I}_E(y,v)|^2 \, dy\, dw \\
&\leq C(K) \times |E|.
\end{aligned}
$$

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