On the rate of convergence to equilibrium in the Becker–Döring equations

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Abstract

We provide a result on the rate of convergence to equilibrium for solutions of the Becker–Döring equations. Our strategy is to use the energy/energy–dissipation relation. The main difficulty is the structure of the equilibria of the Becker–Döring equations, which do not correspond to a gaussian measure, such that a logarithmic Sobolev–inequality is not available. We prove a weaker inequality which still implies for fast decaying data that the solution converges to equilibrium as $e^{-ct^{1/3}}$.

Keywords: Becker–Döring equations, rate of convergence to equilibrium, entropy–dissipation methods

1 Introduction

1.1 The Becker–Döring equations

The Becker–Döring equations are a system of kinetic equations to describe the dynamics of cluster formation in a system with identical particles. They can be used for example to model a variety of phenomena in the kinetics of phase transitions, such as the condensation of liquid droplets in a supersaturated vapor.

In the following clusters are characterized by their size l, which denotes the number of particles in the cluster. The concentration of l–clusters at time

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t will be denoted by $c_l(t)$, and we assume that the clusters are uniformly distributed, such that there is no dependence on a space variable. The main assumption in the Becker–Döring theory is that clusters can change their size only by gaining or shedding one particle. Hence, the rate of change in the concentration of clusters with at least two particles is given by

$$\frac{d}{dt}c_l(t) = J_{l-1}(t) - J_l(t) \qquad \text{for } l \geq 2, \tag{1.1}$$

where J_l denotes the net rate at which l–clusters are converted into $(l+1)$–clusters. We need a different equation for the rate of change of 1–clusters, the free particles, which are also called monomers in the sequel. In the classical Becker–Döring theory [4] the concentration of monomers is just given by a constant. In the following we are however interested in a modified version introduced in [5, 15], where it is assumed that the total density of particles is conserved, i.e.

$$\rho := \sum_{l=1}^{\infty} l c_l(t) \equiv const. \qquad \text{for all } t \geq 0. \tag{1.2}$$

This implies with (1.1) that

$$\frac{d}{dt}c_1(t) = -J_1 - \sum_{l=1}^{\infty} J_l. \tag{1.3}$$

The constitutive relation which gives J_l in terms of c_l is given by

$$J_l(t) = a_l c_1(t) c_l(t) - b_{l+1} c_{l+1}(t), \tag{1.4}$$

with positive kinetic coefficients a_l, b_l which describe the rate at which l–clusters catch and respectively release a monomer.
The Becker–Döring equations are a special case of the so–called discrete coagulation–fragmentation models which have numerous applications in many areas of pure and applied sciences; for an overview of this topic we refer to [8].

Existence of positive solutions of the Becker–Döring equations has been shown in the seminal mathematical paper [3] for data with finite density and coefficients satisfying $a_l = O(l)$. Uniqueness was shown only for a smaller class of coefficients, but more recently the uniqueness result has been extended to a larger class of coefficients in [10]. The main result in [3] is on the convergence of solutions to equilibrium, which is based on exploiting a suitable Lyapunov functional, in physical terms the free energy density. It turns out that equilibrium solutions c^ρ exist for densities $0 \le \rho \le \rho_s$, where ρ_s is the density of saturated vapor. If $\rho \le \rho_s$, then the solution of the Becker–Döring equations converges strongly to c^ρ. If $\rho > \rho_s$, the solution converges weak* to c^{ρ_s} and the excess density $\rho - \rho_s$ corresponds to the formation of larger clusters as time proceeds, i.e. to a phase transition. The existence of metastable states in this case has been established in [13]. It is shown that for moderately small $\rho - \rho_s$ there are data for which the solution stays at least exponentially long in $(\rho - \rho_s)^{-1}$ close to the data before large clusters are formed. Numerical simulations performed in [6] indicate in fact that for generic data the solution always passes through a metastable state. For more details on several aspects of the Becker–Döring equations we also refer to the review article [17].

1.2 The aim of this paper

For the subcritical case $\rho < \rho_s$, metastability has neither been observed nor is it expected. However, to our knowledge there exist no predictions or results on the details of the asymptotic behavior in this case. Even for the related general coagulation–fragmentation models, discrete or continuous, there seems presently only one result available. In [1] it is shown for the continuous pure coagulation equations with constant coefficients that the solution converges exponentially fast to equilibrium. The analysis in [1] seems however not easily extendable to other equations, since it relies on certain exact differential equations satisfied by global quantities.

It is the aim of this paper to provide a result on the speed of convergence to equilibrium of solutions to the Becker–Döring equations in the subcritical case $\rho < \rho_s$. Our strategy to provide an explicit rate of convergence is motivated by so–called entropy– or energy–dissipation methods, which are also one ingredient in [1] and have successfully employed to a variety of problems, in particular in the kinetic theory of gases. In the present situation we face two difficulties. The first lies in the nonlocal structure of the equations, the second, more fundamental, in the structure of the equilibrium solution of the Becker–Döring equation. It has the structure of a general exponential measure, for which so–called logarithmic Sobolev–inequalities do not hold, which are the crucial ingredient in entropy–dissipation methods. Nevertheless, we can prove a weaker inequality, which gives for fast decaying data a rate of convergence to equilibrium as $e^{-ct^{1/3}}$.

In Section 1.3 we first recall in more detail the structure of equilibrium solutions and the results on convergence to equilibrium. Section 1.4 reviews the general idea in entropy–dissipation methods and explains the difficulties we face in the Becker–Döring model. Our main result is given in Section 2 as well as an outline of the main idea of the proof. Finally, the detailed proofs are the content of Section 3.

1.3 Convergence to equilibrium

In order to characterize equilibrium states and to review the results on convergence it is convenient to introduce already at this stage the assumptions on the kinetic coefficients which will be used throughout this paper. We consider a class of coefficients which satisfy the following hypotheses:

(H1) $a_l \geq 1, b_l \geq 1$ for all l.

(H2) $a_l = O(l)$, $b_l = O(l)$.

(H3) Let $Q_1 = 1$ and $b_{l+1}Q_{l+1} = a_l Q_l$ for $l > 1$.
We assume $\lim_{l\to\infty} Q_l^{1/l} = \frac{1}{z_s}$ with $0 < z_s < \infty$.

(H4) $a_l z_s \leq \min(b_l, b_{l+1})$

Typical examples of coefficients which appear e.g. in the theory of phase transitions [13] are

$$a_l = a_1 l^\alpha \qquad \text{for some } 0 < \alpha < 1, \tag{1.5}$$

$$b_l = a_l\Big(z_s + \frac{q}{l^\gamma}\Big) \qquad \text{where } z_s > 0,\ q > 0 \text{ and } 0 < \gamma < 1. \tag{1.6}$$

For example, in three dimensions, if the transport of monomers is dominated by diffusion and clusters are spherical, the typical exponent for coagulation is $\alpha = 1/3$. The Gibbs–Thomson formula gives one obtains $\gamma = 1/3$, z_s is the density of monomers in equilibrium with a flat surface and q is a parameter proportional to the surface tension. We also refer to [14] for a derivation of the coefficients from an Ising model with Kawasaki dynamics. Equilibrium solutions $\bar{c}_l$ are given by the condition

$$J_l = 0 \qquad \text{for } l \geq 1.$$

which implies

$$\bar{c}_l = Q_l z^l, \qquad l \geq 1,$$

where Q_l is defined as in (H3) and $z > 0$ is a parameter. The equilibrium density $\sum_{l=1}^\infty l Q_l z^l$ is bounded for $z < z_s$ due to (H3).

In the following we denote

$$\rho_s := \sum_{l=1}^{\infty} l Q_l z_s^l, \tag{1.7}$$

which might be finite or infinite. This quantity can be interpreted as the density of saturated vapor.
Convergence of solutions to equilibrium under different assumptions on coefficients and data is established in [3, 2, 16] and is based on the fact that there is a Lyapunov functional, the free energy density, which is given by

$$V(c(t)) := \sum_{l=1}^{\infty} c_l \Big(\ln \left(\frac{c_l}{Q_l} \right) - 1 \Big).$$

In fact, it holds

$$\frac{d}{dt} V(c(t)) = - \sum_{l=1}^{\infty} J_l \ln \left(\frac{a_l c_1 c_l}{b_{l+1} c_{l+1}} \right) \leq 0. \tag{1.8}$$

Since V is bounded below, it follows that $J_l \to 0$ as $t \to \infty$ such that $c_l \to Q_l z^l$ for some z. The question remains, what is z and what happens to density conservation (1.2) in the limit as $t \to \infty$. It is shown under some assumptions on coefficients and data in [3, 2, 16] that if $\rho < \rho_s$ then

$$\lim_{t \to \infty} \sum_{l=1}^{\infty} l |c_l(t) - Q_l z^l| \to 0$$

where z is such that $\rho = \sum_{l=1}^{\infty} l Q_l z^l$. If $\rho_s < \infty$ the same holds for $\rho = \rho_s$. However, if $\rho > \rho_s$, we have

$$\lim_{t \to \infty} c_l(t) = Q_l z_s^l \qquad \text{for each } l \geq 1,$$

but the density drops to ρ_s in the limit $t \to \infty$. The so called excess density is contained in larger and larger clusters as times evolves.
For the coefficients satisfying (H1)-(H4), this result has been obtained in [16] for data satisfying $\sum_{l=0}^{\infty} l^2 c_l(0) < \infty$ and $V(c(0)) < \infty$. These assumptions will in particular be satisfied by the data considered in this paper (see (2.1)).

1.4 Entropy dissipation methods

Our strategy to obtain an explicit rate of convergence to equilibrium is inspired by so called entropy–dissipation methods, which have in particular

been developed in the kinetic theory of gases. The advantage of an entropy dissipation method is that it is not necessary to linearize the equation and even for a linear equation it does not require to work in too regular spaces for the solutions.
Let us briefly recall some examples where the method has been successfully employed and compare it to the situation in the Becker–Döring theory. For more details, in particular within the framework of collisional kinetic theory, see also the survey article [20].
The simplest example of the application of such methods is the spatially homogeneous Fokker-Planck equation for the velocity distribution f, i.e.

$$\partial_t f(t,v) = \nabla_v \cdot (\nabla_v f + v f), \quad t \in \mathbb{R}^+,\ v \in \mathbb{R}^d. \tag{1.9}$$

This equation models for instance the dynamics of particles undergoing random collisions over fixed obstacles.
The equilibrium state for Equation (1.9) is the gaussian $M = e^{-v^2/2}$. Hence if f is correctly normalized at the initial time, for instance $\int f(0,v)dv = \int e^{-v^2/2}dv$, then it should converge toward M in latter times.
Equation (1.9) admits a Lyapunov functional, similar to the free energy density for Becker-Döring equations, which is the relative entropy of f with respect to M

$$H(f|M) = \int_{\mathbb{R}^d} f \ \log \frac{f}{M}\, dv. \tag{1.10}$$

This functional satisfies

$$\frac{d}{dt} H(f|M) = -I(f|M) = -\int_{\mathbb{R}^d} f \left| \nabla_v \log \frac{f}{M} \right|^2. \tag{1.11}$$

The right hand side is called the relative Fisher information, for which the following logarithmic Sobolev inequality

$$I(f|M) \geq 2H(f|M) \tag{1.12}$$

holds (see [9] and [18]). This inequality proves that f converges towards M exponentially fast.
The idea here is exactly the same, working with an equivalent form of the free energy density V (compare (2.2) in Section 2).
If we try to adapt the methods used for the Fokker–Planck equation for example, we run into several problems. First, it turns out that the dissipation of the free energy $\frac{d}{dt}V$ controls in fact more the relative entropy with respect to the local equilibrium $(Q_l c_1^l)$ than to $(Q_l z^l)$. (Notice, that $(Q_l c_1^l)$ does not

have density ρ, unless $c_1 = z$.) This forces us to treat the case of large c_1 separately.
The second difficulty is more fundamental. Comparing the expression for the dissipation of the free energy density with equation (1.11), the structure is extremely close: As a matter of fact the expression for $\frac{d}{dt}V$ in (1.8) looks like a discrete version of (1.11). However the real difference comes from the two equilibrium states, a gaussian for Fokker-Planck equation and a sort of modified exponential measure for Becker-Döring.
Now it turns out that the kind of modified logarithmic Sobolev inequality like (1.12) is not true for exponential measures, the limit case being Poisson measure (see the lecture note by M. Ledoux in [12]). This inequality is known only with very strong additional assumptions, typically it would require that the discrete derivative of c_l/c_1^l be uniformly small enough (see [12] again for a continuous version).
This requirement being out of reach here, we prove a weaker inequality. This weaker form still demands some strong uniform bounds on the solution which we also need to prove (more details are given in the next section).
Difficulties in proving corresponding equivalents of (1.12) for different problems are not specific to our situation, they are much harder than here for Boltzmann equation for instance (see [19] in particular). Of course the presence of a space variable would only complicate further everything (we refer to [7] for Fokker-Planck equation). Nevertheless, it is interesting to note that techniques already used in kinetic theory can be successfully applied for coagulation-fragmentation models as e.g. in [11].

2 Rate of convergence to equilibrium

In this paper we are interested in the rate of convergence to equilibrium. We consider fast decaying data with total density smaller than the critical density, i.e. we assume

(H5) $0 < \sum_{l=1}^{\infty} l c_l(0) = \rho < \rho_s$

(H6) For some $\mu > 1$ it holds

$$\sum_{l=1}^{\infty} \mu^l c_l(0) < K_0. \tag{2.1}$$

In the following z will be always such that $\sum_{l=1}^{\infty} l Q_l z^l = \rho$, i.e. $(Q_l z^l)$ is the equilibrium cluster distribution for data satisfying (H5). A critical parameter in the following will be $z_s - z$ and we define

$$\delta = \tfrac{1}{4}(z_s - z).$$

It turns out, that in general we cannot conclude that (2.1) is preserved in time. However, we will establish that there exists $\bar{\mu} = \bar{\mu}(\delta, K_0, \mu)$ with $\bar{\mu} \in (1, \min(\mu, 1 + \frac{\delta}{2z_s}))$ such that $\sum_{l=1}^{\infty} (\bar{\mu})^l c_l(t)$ will be uniformly bounded in time. Notice, that it is natural that $\bar{\mu}$ has to be sufficiently small, since for equilibrium it holds $\sum_{l=1}^{\infty} \mu^l Q_l z^l < \infty$ if $\mu < \frac{z_s}{z}$.
We will also use a different definition of the free energy density which is such that the energy density is always positive and converges to zero as $t \to \infty$. More precisely we write

$$F(c) = \sum_{l=1}^{\infty} c_l \ln\left(\frac{c_l}{Q_l z^l}\right) + Q_l z^l - c_l \tag{2.2}$$

and we will call F in analogy to the examples mentioned in Section 1.4 the relative energy of (c_l) with respect to $(Q_l z^l)$. Notice that

$$F(c) = V(c) - \ln z \sum_{l=1}^{\infty} l c_l + \sum_{l=1}^{\infty} Q_l z^l$$

and due to (1.2) we have $\frac{d}{dt}F = \frac{d}{dt}V$. It is easily seen that assumptions (H3)-(H6) imply $F(c(0)) < \infty$ and hence $F(c(t)) \leq F(c(0)) < \infty$ for all $t > 0$. Our main result shows that F converges exponentially fast, more precisely like $e^{-ct^{1/3}}$, to zero.
All constants in the following results and proofs depend in general on the parameters ρ, a_l, b_l, z_s. We will not explicitly state this dependence. However, we will keep track of the dependence on the parameters δ, μ and K_0.

Theorem 2.1. *Assume that the coefficients a_l, b_l satisfy (H1)-(H4) and consider the solution $c = (c_l)$ of* (1.1) *with data satisfying (H5) and (H6) for some $\mu > 1$.*
Then there exists $c_0 = c_0(K_0, \mu, \delta)$ such that

$$F(c(t)) \leq F(c(0))\, e^{-c_0 t^{1/3}} \tag{2.3}$$

for all $t > 0$.

We do not know whether the decay given by (2.3) is optimal. However, numerical simulations suggest, that for data with $c_1(0) = \rho$ and $c_l(0) = 0$, $l \geq 2$, the convergence can in general not be expected to be of order $e^{-c_0 t}$.
As an immediate consequence of Theorem 2.1 we obtain exponential convergence of the cluster densities in the appropriate norm.

Corollary 2.2. *Let $\eta := \frac{z}{z_s} < 1$. Then there exists a constant C_η such that*

$$\sum_{l=1}^{\infty} l|c_l(t) - Q_l z^l| \leq C_\eta \sqrt{F(c(t))} \tag{2.4}$$

Consequently, under the assumptions of Theorem 2.1 it holds

$$\sum_{l=1}^{\infty} l|c_l - Q_l z^l| \leq C_\eta \, e^{-\frac{c_0}{2} t^{1/3}}. \tag{2.5}$$

Let us give a brief overview of the main steps and ideas for the proof of Theorem 2.1.
For that and the upcoming analysis, we first recall the notation for the relative energy

$$F(c) = \sum_{l=1}^{\infty} c_l \ln\left(\frac{c_l}{Q_l z^l}\right) + Q_l z^l - c_l =: \sum_{l=1}^{\infty} Q_l z^l f\Big(\frac{c_l - Q_l z^l}{Q_l z^l}\Big)$$

with $f(z) := (1+z)\ln(1+z) - z \geq 0$ and denote by

$$D := -\frac{d}{dt}F = \sum_{l=1}^{\infty} (a_l c_1 c_l - b_{l+1} c_{l+1}) \ln\left(\frac{a_l c_1 c_l}{b_{l+1} c_{l+1}}\right).$$

the energy–dissipation rate.
To prove Theorem 2.1 we need to find a lower bound on the dissipation rate D. Here, we have to differentiate between two situations: first, when c_1 is large, i.e. $c_1(t) \geq z_s - \delta$, and second, when $c_1 \leq z_s - \delta/2$, a case, which we also call subcritical from now on. In the first case, we prove (cf. Lemma 3.6, Section 3.2) that whenever $c_1 \geq z_s - \delta$, then

$$D \geq \frac{\delta^4}{C}. \tag{2.6}$$

The idea of the proof is simple: if c_1 is large, then not too many elements of the sum defining D can be small, since then the constraint $\rho = \sum_{l=1}^{\infty} l c_l$ cannot be satisfied. The proof of (2.6) is independent of a bound on $\sum_{l=1}^{\infty} \mu^l c_l$ and only requires $\sum_{l=1}^{\infty} l c_l = \rho < \infty$.
The main part of the proof of Theorem 2.1 is then, to find a lower bound on the dissipation rate when $c_1 \leq z_s - \delta/2$ (Proposition 3.7, Section 3.3). Here, the key idea is, that D controls the relative energy of (c_l) with respect to $(Q_l c_1^l)$ which again dominates F. However, this is possible only if $0 < c_1 <$

z_s, hence the restriction of this idea to the subcritical case. The proof of Proposition 3.7 is split in several Lemmas in Section 3.3. It is shown that

$$D \geq \frac{1}{C|\ln F|^2} F.$$

which then by a simple ODE argument gives (2.3). For the main estimate, contained in Lemma 3.10 in Section 3.4, it is essential to know that

$$\sum_{l=1}^{\infty} (\bar{\mu})^l c_l(t) \leq C$$

uniformly in time for some $\bar{\mu} > 1$. This will be a consequence of (2.1). However, as pointed out before, we cannot show that (2.1) is preserved in time. This is only true over time intervals where c_1 is subcritical. Over time intervals, where c_1 is large, we can however construct a smaller μ' such that $\sum_{l=1}^{\infty} (\mu')^l c_l$ is bounded. Similarly, we have to keep c_1 uniformly bounded away from zero (after possibly an initial time interval, in case $c_1(0) = 0$). The corresponding a–priori estimates to control $\sum_{l=1}^{\infty} (\mu')^l c_l$ from above and c_1 from below, together with the proof of Corollary 2.2, are the content of Section 3.1.
The precise argument, how to combine all ingredients to a proof of Theorem 2.1 is given in Section 3.4

3 The proofs

3.1 A–priori estimates

In the following, C will always denote a constant, which may change from line to line, and which may depend on the parameters ρ, a_l, b_l. Dependence on the parameters δ, K_0, μ etc. will however be indicated by the notation $C = C(\delta)$ etc.
We first show (2.4) in Corollary 2.2 which is a consequence of the convexity of f.

Lemma 3.1. *Let $\eta := \frac{z}{z_s} < 1$. Then there exists a constant C_η such that*

$$\sum_{l=1}^{\infty} l|c_l - Q_l z^l| \leq \max\left(2F(c), C_\eta \sqrt{F(c)}\right)$$

Proof. We use the relation

$$yx \le f(x) + f^*(y) \tag{3.1}$$

where f^* is the dual of f and is given by

$$f^*(y) = e^y - y - 1. \tag{3.2}$$

Notice that f and f^* satisfy

$$f(|x|) \le f(x) \qquad \text{and} \qquad f^*(ry) \le r^2 f^*(y) \qquad \text{for } r \in [0,1]. \tag{3.3}$$

With $y = \varepsilon \frac{1}{2} \ln(\frac{1}{\eta}) l$ for some $\varepsilon, \eta \in (0,1]$ and $x = \frac{|c_l - Q_l z^l|}{Q_l z^l}$ we find with (3.1), (3.2) and (3.3) that

$$\varepsilon \tfrac{1}{2} \ln\left(\frac{1}{\eta}\right) l \, \frac{|c_l - Q_l z^l|}{Q_l z^l} \le \varepsilon^2 \exp\left\{ \tfrac{1}{2} \ln\left(\frac{1}{\eta}\right) l \right\} + f\left(\frac{c_l - Q_l z^l}{Q_l z^l}\right).$$

If we multiply with $Q_l z^l$, sum over $l \ge 1$ and use that due to (H3) it holds $Q_l z^l \approx \exp\{-\ln\left(\frac{1}{\eta}\right) l\}$ for large l, we find

$$\begin{aligned} \tfrac{1}{2} \ln\left(\frac{1}{\eta}\right) \sum_{l=1}^{\infty} l |c_l - c_l^s| &\le C\varepsilon \sum_{l=1}^{\infty} \exp\left\{ -\tfrac{1}{2} \ln\left(\frac{1}{\eta}\right) l \right\} + \frac{1}{\varepsilon} F(c) \\ &:= C_\eta \varepsilon + \frac{1}{\varepsilon} F(c). \end{aligned}$$

Choosing $\varepsilon = 1$ if $F \ge C_\eta$ and $\varepsilon = \sqrt{F(c)/C_\eta}$ otherwise finishes the proof of the lemma. □

In the next lemma we show, that if $\sum_{l=1}^{\infty} \mu^l c_l(t_1) < \infty$, one can find for any finite time interval (t_1, t_2) a μ' such that $\sum_{l=1}^{\infty} (\mu')^l c_l(t_2) < \infty$.

Lemma 3.2. *Let $[t_1, t_2) \subset [0, \infty)$ be an arbitrary finite time interval and assume that for some $\mu > 1$ it holds*

$$\sum_{l=1}^{\infty} \mu^l c_l(t_1) =: M_1 < \infty.$$

Then it holds with

$$\mu' = 1 + e^{-C(t_2 - t_1)}(\mu - 1)$$

that

$$\sum_{l=1}^{\infty} (\mu')^l c_l(t_2) \le C(t_2 - t_1) + M_1.$$

Proof. Let us first present the formal argument for the proof. We compute

$$\begin{aligned}\frac{d}{dt}\sum_{l=2}^{\infty}\mu^l c_l &= \sum_{l=2}^{\infty}\mu^l(J_{l-1}-J_l)\\ &= \sum_{l=2}^{\infty}(\mu^{l+1}-\mu^l)J_l \,+\, \mu^2 J_1\\ &\le (\mu-1)\sum_{l=2}^{\infty}\mu^l a_l c_1 c_l \,+\, \mu^2 a_1 c_1^2\\ &\le C(\mu-1)\sum_{l=2}^{\infty}\mu^l l c_l + C\mu^2.\end{aligned} \tag{3.4}$$

Now we define

$$F(t,\mu) := \sum_{l=2}^{\infty}\mu^l c_l.$$

Then (3.4) implies that F satisfies

$$\partial_t F \le C(\mu(\mu-1)\partial_\mu F + \mu^2).$$

We can assume that $\mu \in (1,2)$ and hence we have

$$\partial_t F \le C((\mu-1)\partial_\mu F + 1).$$

We define now the corresponding characteristics

$$X(t,\mu) = 1 + e^{-C(t-t_1)}(\mu-1)$$

and obtain

$$F(t, X(t,\mu)) \le C(t-t_1) + F(t_1,\mu).$$

Hence, if we choose $\mu' = 1 + e^{-C(t_2-t_1)}\mu$ we find

$$F(t_2,\mu') \le C(t_2-t_1) + F(t_1,\mu),$$

which finishes the proof, if all manipulations can indeed be performed.
In order to prove the estimate rigorously, we can proceed as in [13] for

example. For that we introduce an auxiliary finite system $(c_l^{(n)})$, such that

$$\begin{aligned}\frac{d}{dt}c_l^{(n)} &= J_{l-1}^{(n)} - J_l^{(n)}, \qquad 2 \le l \le n,\\ J_l^{(n)} &= a_l c_1^{(n)} c_l^{(n)} - b_{l+1} c_{l+1}^{(n)},\\ c_l^{(n)} &= 0, \qquad l \ge n+1,\\ \frac{d}{dt}c_1^{(n)} &= -J_1^{(n)} - \sum_{l=1}^{n} J_l^{(n)},\\ c_l^{(n)}(t_1) &= c_l(t_1), \qquad l \le n.\end{aligned}$$

Notice, that it holds

$$J_n^{(n)} = a_l c_1^{(n)} c_n^{(n)} \ge 0, \tag{3.5}$$

since a solution of this finite system also satisfies $c_l^{(n)} \ge 0$. Now we find

$$\begin{aligned}\frac{d}{dt}\sum_{l=2}^{n} \mu^l c_l^{(n)} &= (\mu-1)\sum_{l=2}^{n} \mu^l J_l^{(n)} \;+\; \mu^2 J_1^{(n)} - \mu^{(n+1)} J_n^{(n)}\\ &\le (\mu-1)\sum_{l=2}^{n} \mu^l J_l^{(n)} \;+\; C\\ &\le C(\mu-1)\sum_{l=2}^{n} \mu^l l c_l \;+\; C.\end{aligned}$$

With

$$F^{(n)}(t,\mu) := \sum_{l=2}^{n} \mu^l c_l^{(n)}$$

we can proceed as described above to conclude

$$F^{(n)}(t, X(t,\mu)) \le C(t-t_1) \;+\; F^{(n)}(t_1,\mu).$$

Now we let $n \to \infty$. It is straightforward and described also in [13], to conclude that $c_l^{(n)} \to c_l$, which is the unique solution of the Becker–Döring equations for data $(c_l(t_1))$. Since $F^{(n)}(t_1,\mu) \to F(t_1,\mu) < \infty$, the conclusion of the lemma follows. □

Lemma 3.3. *Let* $[t_1, t_2) \subset [0,\infty)$ *be an arbitrary (possibly infinite) time interval such that*

$$c_1 \le z_s - \delta/2 \qquad \textit{for all } t \in (t_1, t_2).$$

Furthermore assume that

$$\sum_{l=1}^{\infty} \mu^l c_l(t_1) =: M_1 < \infty$$

for some $1 < \mu \leq 1 + \frac{\delta}{4z_s}$. *Then it holds*

$$\sup_{t \in (t_1,t_2)} \sum_{l=1}^{\infty} \mu^l c_l(t) \leq \frac{C}{\delta(\mu-1)} + M_1$$

Proof. We present the formal computations. For a rigorous proof one may proceed as described in the proof of Lemma 3.2 and we omit the details. We now write

$$\begin{aligned}
\frac{d}{dt}\sum_{l=2}^{\infty} \mu^l c_l &= \sum_{l=2}^{\infty} \mu^l (J_{l-1} - J_l) \\
&= (\mu - 1)\sum_{l=1}^{\infty} \mu^l J_l \; + \; \mu J_1 \\
&= (\mu - 1)\sum_{l=1}^{\infty} (\mu^l a_l c_1 c_l - \mu^{l-1} b_l c_l) \; + \; (\mu-1) b_1 c_1 \; + \; \mu J_1 \\
&= (\mu - 1)\sum_{l=1}^{\infty} \mu^{l-1} a_l \Big(\mu c_1 - \frac{b_l}{a_l}\Big) c_l + \; (\mu-1) b_1 c_1 \; + \; \mu J_1.
\end{aligned}$$

Now we use (H4), i.e. $z_s a_l \leq b_l$, that with $\mu \leq 1 + \frac{\delta}{4z_s}$ it holds $z_s - \mu c_1 \geq \delta/2$, and that $\sum_{l=1}^{\infty} \mu^l a_l c_l \geq \sum_{l=1}^{\infty} \mu^l c_l$ due to (H1), to find

$$\frac{d}{dt}\sum_{l=2}^{\infty} \mu^l c_l \leq -\delta \frac{\mu-1}{2\mu} \sum_{l=1}^{\infty} \mu^l c_l \; + \; b_1 c_1 + 2 a_1 c_1^2.$$

Since $b_1 c_1 + a_1 c_1^2 \leq b_1 \rho + a_1 \rho^2 \leq C$ the desired result follows. □

In the next lemma we show that c_1 is positive after a possible initial time layer.

Lemma 3.4. *There exists* $\tilde{\delta} = \tilde{\delta}(K_0)$ *such that for all* $t \geq 1$ *we have*

$$c_1 \geq \tilde{\delta} e^{-\rho t}.$$

Proof. We compute

$$\begin{aligned}
\partial_t c_1 &= -J_1 - \sum_{l=1}^{\infty} J_l \\
&\geq -a_1 c_1^2 \,-\, c_1 \sum_{l=1}^{\infty} a_l c_l \,+\, \sum_{l=1}^{\infty} b_{l+1} c_{l+1} \\
&= -a_1 c_1^2 \,-\, \sum_{l=1}^{\infty} (a_l c_1 - b_l) c_l \,-\, b_1 c_1 \\
&\geq -a_1 c_1^2 + (z_s - c_1) \sum_{l=1}^{\infty} a_l c_l \,-\, b_1 c_1
\end{aligned}$$

where we used (H4) in the last inequality. Now

$$\rho^2 \leq \Big(\sum_{l=1}^{\infty} a_l c_l \Big) \Big(\sum_{l=1}^{\infty} l^2 c_l \Big).$$

We find

$$\frac{d}{dt} \sum_{l=2}^{\infty} l^2 c_l \leq \rho \sum_{l=2}^{\infty} l^2 c_l$$

and hence

$$\sum_{l=1}^{\infty} l^2 c_l(t) \leq C K_0 e^{\rho t}.$$

Thus

$$\sum_{l=1}^{\infty} a_l c_l \geq \frac{\rho^2}{C K_0} e^{-\rho t}$$

and as long as $c_1 \leq \tilde{\delta} e^{-\rho t}$, with $\tilde{\delta} = \frac{1}{CK_0}$ for some C, we have

$$\partial_t c_1 \geq -(a_1 + b_1) c_1 + (z_s - c_1) \frac{\rho^2}{C K_0} e^{-\rho t} \geq \tilde{\delta} e^{-\rho t}$$

and the result follows. □

Lemma 3.5. *Let* $[t_1, t_2) \subset [0, \infty)$ *be an arbitrary (possibly infinite) time interval such that*

$$c_1 - z_s \leq \delta/2 \qquad \text{for all } t \in (t_1, t_2)$$

and

$$\sup_{t\in(t_1,t_2)} \sum_{l=1}^{\infty} \mu^l c_l(t) \le M_2.$$

Then, if $c_1(t_1) \ge \delta_1$ *with* $\delta_1 = \frac{1}{CM_2}$ *it holds*

$$\inf_{t\in(t_1,t_2)} c_1(t) \ge \delta_1.$$

Proof. The proof is analogous to the proof of Lemma 3.4, with the difference that we have now a uniform bound

$$\sum_{l=1}^{\infty} a_l c_l \ge \frac{\rho^2}{CM_2}.$$

□

3.2 Decay of the energy when c_1 is large

The following lemma provides a uniform estimate for the rate of decay of the energy if $c_1 \ge z_s - \delta$.

Lemma 3.6. *There exists a constant* C*, such that for all* t *with* $c_1(t) \ge z_s - \delta$ *it holds*

$$\frac{d}{dt}F(c(t)) + \frac{\delta^4}{C} \le 0.$$

Proof. We recall the expression of the dissipation rate and $b_l \ge 1$ to find

$$\begin{aligned} D &= \sum_{l=1}^{\infty} (a_l c_1 c_l - b_{l+1} c_{l+1}) \ln\left(\frac{a_l c_1 c_l}{b_{l+1} c_{l+1}}\right) \\ &\ge \sum_{l=1}^{\infty} \left(\frac{a_l c_1}{b_{l+1}} c_l - c_{l+1}\right) \ln\left(\frac{a_l c_1 c_l}{b_{l+1} c_{l+1}}\right). \end{aligned}$$

Now we choose a real number $\lambda < 1$ such that $z_s - 2\delta < \lambda c_1 < z_s$. We denote by l_0 the first index l such that

$$c_{l+1} \le \lambda \frac{a_l c_1}{b_{l+1}} c_l. \tag{3.6}$$

This number necessarily exists, since otherwise we would have

$$\rho = \sum_{l=1}^{\infty} l c_l \ge c_1 + \sum_{l=2}^{\infty} l(\lambda c_1)^{l-1} Q_l c_1 > \sum_{l=1}^{\infty} l Q_l z^l = \rho.$$

Then we have

$$D \geq (1-\lambda)\ln\frac{1}{\lambda}\frac{a_{l_0}c_1}{b_{l_0+1}}c_{l_0} \geq (1-\lambda)\ln\frac{1}{\lambda}\,(\lambda c_1)^{l_0}\,Q_{l_0}c_1.$$

The principle idea to estimate the right hand side is quite simple. If l_0 is too large, then $c_{l+1} > \lambda\frac{a_l c_1 c_l}{b_{l+1}}$ for too many indices, and the total density would be larger than ρ.
We define $\lambda' = \lambda(1-\frac{\delta}{z_s})$ such that $z_s - 3\delta < \lambda' c_1$ and

$$\tilde{\rho} = \sum_{l=1}^{\infty} l(\lambda' c_1)^{l-1}\,Q_l c_1.$$

By the definition of Q_l, although the series defining $\tilde{\rho}$ is not exactly geometric, there exists a constant $C > 0$ depending only on a_l and b_l and not on l_0, c_1 or λ' such that

$$\sum_{l\geq l_0+1} l\,(\lambda' c_1)^{l-1}\,Q_l c_1 \leq C l_0\,(\lambda' c_1)^{l_0}\,Q_{l_0}.$$

Then we obtain

$$C l_0(\lambda' c_1)^{l_0}\,Q_{l_0} \geq \tilde{\rho} - \sum_{l=1}^{l_0} l\,(\lambda' c_1)^{l-1}\,Q_l c_1 \geq \tilde{\rho} - \sum_{l=1}^{l_0} l\,c_l \geq \tilde{\rho} - \rho,$$

because up to the index l_0, we have

$$c_l \geq (\lambda c_1)^{l-1}\,Q_l c_1 \geq (\lambda' c_1)^{l-1}\,Q_l c_1.$$

We estimate the difference $\tilde{\rho} - \rho$ by

$$\begin{aligned}\tilde{\rho} - \rho &= \sum_{l=1}^{\infty} lQ_l((\lambda' c_1)^{l-1}c_1 - z^l)\\ &\geq \sum_{l=1}^{\infty} lQ_l((z+\delta)^l - z^l)\\ &\geq \delta\sum_{l=1}^{\infty} lQ_l z^{l-1} = \delta\frac{\rho}{z}.\end{aligned}$$

Furthermore we easily check that it holds for all l and in particular l_0

$$l(\lambda' c_1)^l \leq \frac{z_s}{\delta}(\lambda c_1)^l.$$

Hence, gathering all the estimates, we have

$$\begin{aligned} D &\geq (1-\lambda)\ln\frac{1}{\lambda}\,(\lambda c_1)^{l_0}Q_{l_0}c_1 \\ &\geq (1-\lambda)\ln\frac{1}{\lambda}\frac{\delta}{z_s}\,l_0(\lambda' c_1)^{l_0}Q_{l_0}c_1 \\ &\geq \frac{1}{C}(1-\lambda)\ln\frac{1}{\lambda}\frac{\delta}{z_s}\,\delta\frac{\rho}{z}(z_s-\delta) \\ &\geq \frac{\delta^4}{C}, \end{aligned}$$

which proves the lemma. □

3.3 Decay of the energy when c_1 is subcritical

Proposition 3.7. *Let $[t_1, t_2) \subset [0, \infty)$ be an arbitrary time interval and assume*

$$c_1(t) \leq z_s - \delta/2 \qquad \textit{for all } t \in [t_1, t_2),$$

as well as

$$\sum_{l=1}^{\infty} \mu^l c_l(t_1) \leq M_1 < \infty,$$

and

$$\delta_1 \leq c_1(t_1)$$

with sufficiently small $\delta_1 = \delta_1(M_1, \mu)$.
Then there exists $c_0 = c_0(M_1, \delta, \mu)$ such that

$$F(c(t)) \leq F(c(t_1))e^{-c_0(t-t_1)^{1/3}}$$

for all $t \in [t_1, t_2)$.

We first recall that due to Lemmas 3.3 and 3.5 it holds under the assumptions of Proposition 3.7 that

$$\sup_{t\in(t_1,t_2)} \sum_{l=1}^{\infty} \mu^l c_l(t) \leq M_1 + C(\delta, \mu) \tag{3.7}$$

and

$$\inf_{t\in(t_1,t_2)} c_1(t) \geq \delta_1.$$

An important role in the following will be played by the relative energy of (c_l) with respect to $(Q_l c_1^l)$

$$F_1(c) := \sum_{l=1}^{\infty} c_l \ln\left(\frac{c_l}{Q_l c_1^l}\right) + Q_l c_1^l - c_l = \sum_{l=1}^{\infty} Q_l c_1^l f\Big(\frac{c_l - Q_l c_1^l}{Q_l c_1^l}\Big).$$

We will see that $F_1 < \infty$ if $c_1 < z_s$.

Lemma 3.8. *If* $0 < c_1 < z_s$ *we have*

$$F(c) \leq F_1(c) < \infty$$

Proof. First, assume that all the sums are absolutely convergent. In view of

$$F_1 = \sum_{l=1}^{\infty} Q_l z^l f\Big(\frac{c_l - Q_l z^l}{Q_l z^l}\Big) \; + \sum_{l=1}^{\infty} c_l \ln\left(\frac{Q_l z^l}{Q_l c_1^l}\right) + Q_l c_1^l - Q_l z^l.$$

we find, due to $\sum_{l=1}^{\infty} l c_l = \sum_{l=1}^{\infty} l Q_l z^l$, that

$$\begin{aligned}
\sum_{l=1}^{\infty} c_l \ln\left(\frac{Q_l z^l}{Q_l c_1^l}\right) &= \sum_{l=1}^{\infty} l c_l \ln\left(\frac{z}{c_1}\right) \\
&= \sum_{l=1}^{\infty} l Q_l z^l \ln\left(\frac{z}{c_1}\right) \\
&= \sum_{l=1}^{\infty} Q_l z^l \ln\left(\frac{Q_l z^l}{Q_l c_1^l}\right),
\end{aligned}$$

and thus

$$F_1 = F \; + \sum_{l=1}^{\infty} Q_l c_1^l \, f\Big(\frac{Q_l z^l - Q_l c_1^l}{Q_l c_1^l}\Big) \geq F.$$

Since

$$\sum_{l=1}^{\infty} Q_l c_1^l \, f\Big(\frac{Q_l z^l - Q_l c_1^l}{Q_l c_1^l}\Big) = \ln\left(\frac{z}{c_1}\right)\rho + \sum_{l=1}^{\infty} Q_l (c_1^l - z^l) < \infty$$

if $0 < c_1 < z_s$, we find that F is indeed finite under this assumption. $\square$

We denote in the following

$$u_l := \frac{c_l}{Q_l c_1^l}.$$

All estimates which follow will be pointwise in time, so for convenience we omit the dependence on t in the notation.

Lemma 3.9.

$$F_1 = \sum_{l=1}^{\infty} Q_l c_1^l (u_l \ln u_l + 1 - u_l) \le \frac{1}{z_s - c_1} \sum_{l=1}^{\infty} Q_l c_1^{l+1} (u_{l+1} - u_l) \ln u_{l+1}.$$

Proof. Step 1:(“Integration by parts”) It holds for $r < 1$ and functions $\Phi : \mathbb{N} \to \mathbb{R}$ such that $\sum_{l=1}^{\infty} r^l \Phi(l) < \infty$, that

$$\sum_{l=1}^{\infty} r^l \Phi(l) = \frac{r}{1-r} \sum_{l=1}^{\infty} r^l (\Phi(l+1) - \Phi(l)) \; + \; \frac{r}{1-r} \Phi(1). \tag{3.8}$$

Equation (3.8) follows easily via

$$\begin{aligned} \sum_{l=1}^{\infty} r^l (\Phi(l+1) - \Phi(l)) &= \sum_{l=2}^{\infty} r^{l-1} \Phi(l) - \sum_{l=1}^{\infty} r^l \Phi(l) \\ &= \sum_{l=1}^{\infty} r^{l-1} (1-r) \Phi(l) \; - \; \Phi(1) \\ &= \frac{1-r}{r} \sum_{l=1}^{\infty} r^l \Phi(l) \; - \; \Phi(1). \end{aligned}$$

Step 2: We first observe that with

$$r = \frac{c_1}{z_s} \qquad \text{and} \qquad \Phi(l) = z_s^l Q_l (u_l \ln u_l + 1 - u_l)$$

it holds

$$\sum_{l=1}^{\infty} r^l \Phi(l) = F_1 < \infty.$$

Now we can employ (3.8) to find

$$\begin{aligned} F_1 = & \frac{c_1}{z_s - c_1} \sum_{l=1}^{\infty} \frac{c_1^l}{z_s^l} \Big(z_s^{l+1} Q_{l+1} (u_{l+1} \ln u_{l+1} + 1 - u_{l+1}) \\ & - Q_l z_s^l (u_l \ln u_l + 1 - u_l) \Big) \\ = & \frac{c_1}{z_s - c_1} \sum_{l=1}^{\infty} c_1^l Q_l (u_{l+1} - u_l) \ln u_{l+1} \\ & + \frac{c_1}{z_s - c_1} \sum_{l=1}^{\infty} \frac{c_1^l}{z_s^l} \Big((z_s^{l+1} Q_{l+1} - z_s^l Q_l)(u_{l+1} \ln u_{l+1} + 1 - u_{l+1}) \\ & - z_s^l Q_l \Big(u_l \ln \Big(\frac{u_l}{u_{l+1}} \Big) + u_{l+1} - u_l \Big) \Big). \end{aligned}$$

While the last term is always negative, the second term is negative if $z_s Q_{l+1} \leq Q_l$. But this holds due to (H4) and thus

$$F_1 \leq \frac{c_1}{z_s - c_1} \sum_{l=1}^{\infty} c_1^l Q_l (u_{l+1} - u_l) \ln u_{l+1}.$$

□

Now we are in the position to prove the main estimate.

Lemma 3.10. *There exists a constant $C = C(M_1, \mu, \delta)$ such that*

$$F_1 \leq CD \Big| \ln\left(\frac{1}{F}\right) \Big|^2 + \tfrac{1}{2} F$$

Proof. Step 1: We first notice that with $(x-y)\ln\left(\frac{x}{y}\right) \geq \frac{(x-y)^2}{\max(x,y)}$ and (H1) it holds

$$\begin{aligned} D &= \sum_{l=1}^{\infty} c_1^{l+1} Q_l a_l (u_{l+1} - u_l) \ln\left(\frac{u_{l+1}}{u_l}\right) \\ &\geq \sum_{l=1}^{\infty} c_1^{l+1} Q_l a_l \frac{(u_{l+1} - u_l)^2}{\max(u_l, u_{l+1})} \\ &\geq \sum_{l=1}^{\infty} c_l^{l+1} Q_l \frac{(u_{l+1} - u_l)^2}{\max(u_l, u_{l+1})}. \end{aligned} \tag{3.9}$$

Thus

$$\sum_{l=1}^{\infty} c_1^{l+1} Q_l (u_{l+1} - u_l) \ln u_{l+1} \leq D + \sum_{l=1}^{\infty} c_1^{l+1} Q_l (u_{l+1} - u_l) \ln u_l$$

and we note that the second term on the right hand side is positive only if

- Case I: $u_{l+1} \leq u_l \leq 1$
- Case II: $u_{l+1} \geq u_l \geq 1$.

Step 2: (Case I)
Let $I := \{l \,|\, u_{l+1} \leq u_l \leq 1\}$. Then

$$\begin{aligned} &\sum_{l \in I} c_1^{l+1} Q_l (u_{l+1} - u_l) \ln u_l \\ &\qquad \leq \left(\sum_{l \in I} c_1^{l+1} Q_l \frac{(u_{l+1} - u_l)^2}{u_l} \right)^{1/2} \left(\sum_{l \in I} c_1^{l+1} Q_l u_l |\ln u_l|^2 \right)^{1/2}. \end{aligned}$$

Now, with (3.9)

$$\sum_{l\in I} c_1^{l+1} Q_l \frac{(u_{l+1}-u_l)^2}{u_l} \le \sum_{l\in I} c_1^{l+1} Q_l \frac{(u_{l+1}-u_l)^2}{\max(u_l,u_{l+1})} \le D.$$

Furthermore one easily checks that

$$x|\ln x|^2 \le 2(x\ln x + 1 - x) \qquad \text{for } x\in[0,1].$$

Hence, it follows

$$\sum_{l\in I} c_1^{l+1} Q_l |\ln u_l|^2 \le 2c_1F_1.$$

and we obtain

$$\frac{1}{z_s-c_1}\sum_{l\in I} c_1^{l+1} Q_l (u_{l+1}-u_l)\ln u_{l+1} \le \frac{1}{z_s-c_1}(D+\sqrt{D}\sqrt{2c_1F_1}). \quad (3.10)$$

¿From now on we consider case II:
Step 3: We first note that

$$\ln x \le 2\Big(\frac{x-1}{x}\ln(1+x)\Big) \qquad \text{for } x\ge 1.$$

Thus,

$$\begin{aligned}&\sum_{1\le u_l\le u_{l+1}} c_1^{l+1} Q_l (u_{l+1}-u_l)\ln u_{l+1}\\ &\qquad \le 2\sum_{1\le u_l\le u_{l+1}} c_1^{l+1} Q_l (u_{l+1}-u_l)\frac{u_{l+1}-1}{u_{l+1}}\ln(1+u_{l+1}).\end{aligned}$$

Step 4: For some $\lambda \gg 1$ to be determined we split the sum

$$\sum_{1\le u_l\le u_{l+1}} c_1^{l+1} Q_l (u_{l+1}-u_l)\frac{(u_{l+1}-1)}{u_{l+1}}\ln(1+u_{l+1}) = \sum_{u_{l+1}>\lambda} ... + \sum_{1\le u_l\le u_{l+1}\le\lambda} ...$$

Step 5: (The case $1 \le u_l \le u_{l+1} \le \lambda$)

$$\begin{aligned}
\sum_{1\le u_l\le u_{l+1}\le\lambda} & c_1^{l+1} Q_l (u_{l+1}-u_l)\frac{u_{l+1}-1}{u_{l+1}}\ln(1+u_{l+1}) \\
&\le \Bigg(\sum_{u_{l+1}\ge u_l} c_1^{l+1} Q_l \frac{(u_{l+1}-u_l)^2}{u_{l+1}}\Bigg)^{1/2} \\
&\quad\cdot \Bigg(\sum_{1\le u_{l+1}\le\lambda} c_1^{l+1} Q_l \frac{(u_{l+1}-1)^2}{u_{l+1}} |\ln(1+u_{l+1})|^2\Bigg)^{1/2} \\
&\le 2\ln(1+\lambda)\sqrt{D}\Bigg(\sum_{1\le u_{l+1}\le\lambda} c_1^{l+1} Q_l \frac{(u_{l+1}-1)^2}{u_{l+1}}\Bigg)^{1/2}.
\end{aligned}$$

We observe

$$\frac{|x-1|^2}{x} \le x\ln x + 1 - x \qquad \text{for } x\ge 1.$$

Hence

$$\sum_{1\le u_{l+1}\le\lambda} c_1^{l+1} Q_l (u_{l+1}-u_l)\frac{u_{l+1}-1}{u_{l+1}}\ln(1+u_{l+1}) \le \ln(1+\lambda)\sqrt{D}\sqrt{F_1}.$$

Step 6: (The case $u_{l+1} > \lambda$)
For some small constant $\varepsilon > 0$ we have

$$\begin{aligned}
2\sum_{u_{l+1}>\lambda} & c_1^{l+1} Q_l (u_{l+1}-u_l)\frac{u_{l+1}-1}{u_{l+1}}\ln(1+u_{l+1}) \\
&\le 2\sum_{u_{l+1}>\lambda} c_1^{l+1} Q_l (u_{l+1}+1)\ln(1+u_{l+1}) \\
&\le C\sum_{u_{l+1}>\lambda} c_1^{l+1} Q_l |u_{l+1}|^{1+2\varepsilon} \\
&\le \frac{C}{\lambda^\varepsilon}\sum_{u_{l+1}>\lambda} c_1^{l+1} Q_l |u_{l+1}|^{1+\varepsilon} \\
&= \frac{C}{\lambda^\varepsilon}\sum_{u_{l+1}>\lambda} c_1^{l+1} Q_l \frac{|c_{l+1}|^{1+\varepsilon}}{|Q_{l+1}c_1^{l+1}|^{1+\varepsilon}} \\
&\le \frac{C}{\lambda^\varepsilon}\sum_{l=1}^{\infty} c_{l+1}\frac{b_{l+1}}{a_l}\frac{|c_{l+1}|^\varepsilon}{\left|Q_{l+1}^{\varepsilon/(l+1)} c_1^\varepsilon\right|^{(l+1)}}.
\end{aligned}$$

Thus, if

$$\left(\frac{b_{l+1}}{a_l}\right)^{1/(l+1)} \frac{1}{|Q_{l+1}^{\varepsilon/(l+1)} c_1^{\varepsilon}|} \le \mu \tag{3.11}$$

with μ as in (3.7), then we find

$$2 \sum_{u_{l+1}>\lambda} c_1^{l+1} Q_l (u_{l+1} - u_l) \frac{u_{l+1}-1}{u_{l+1}} \ln(1+u_{l+1}) \le \frac{C}{\lambda^{\varepsilon}}.$$

Since by (H4) we have $\left(\frac{b_{l+1}}{a_l}\right)^{1/(l+1)} \to 1$ and $Q_l^{1/l} \to 1/z_s$ inequality (3.11) follows if

$$\varepsilon < \frac{\ln \mu}{\ln(\frac{z_s}{\delta_1})}.$$

Step 7: (Summary) We summarize Steps 1-6 and obtain

$$\begin{aligned} F_1 &\le \frac{1}{z_s - c_1}\Big(2D + \sqrt{D}\sqrt{2c_1 F_1} + 2\ln(1+\lambda)\sqrt{D}\sqrt{F_1} + \frac{C}{\lambda^{\varepsilon}}\Big) \\ &\le \frac{1}{2}F_1 + C\Big(D + |\ln\lambda|^2 D + \frac{1}{\lambda^{\varepsilon}}\Big) \end{aligned}$$

We choose

$$\lambda = \left(\frac{4C}{F}\right)^{1/\varepsilon}$$

and note that

$$\ln \frac{1}{F^{1/\varepsilon}} \le \frac{1}{\varepsilon} \ln \frac{1}{F}.$$

Summarized we find for $C = C(M_1, \mu, \delta)$ that

$$F_1 \le C \Big| \ln\left(\frac{1}{F}\right)\Big|^2 D + \tfrac{1}{2}F$$

which finishes the proof of the lemma. □

With Lemma 3.8 and Lemma 3.10 we have

$$F \le C\, D \Big| \ln \frac{1}{F}\Big|^2,$$

or

$$\frac{d}{dt}F + c_0 \frac{F}{|\ln \frac{1}{F}|^2} \le 0$$

for $c_0 = \frac{1}{C} > 0$. This is equivalent to

$$-\frac{d}{dt}\frac{1}{3}\Big(\ln\frac{1}{F}\Big)^3 \le -c_0 t$$

which finishes the proof of Proposition 3.7.

3.4 Proof of Theorem 2.1

We can now summarize the results to proof Theorem 2.1.
First, we notice that we cannot conclude that once c_1 is below $z_s - \delta/2$, that it will stay subcritical, unless the energy is already sufficiently small.
Thus, let (t_n^-, t_n^+), $n = 1, 2, \dots$ be successive disjoint time intervals such that

$$\begin{aligned} c_1(t_n^-) &\leq z_s - \delta, \\ c_1(t_n^+) &\leq z_s - \delta, \\ c_1(t) &\geq z_s - \delta \qquad \text{for all } t \in (t_n^-, t_n^+), \\ c_1(t) &\geq z_s - \delta/2 \qquad \text{at least for one } t \in (t_n^-, t_n^+). \end{aligned}$$

If no such interval exists, then it holds $c_1(t) \leq z_s - \delta/2$ for all $t \geq 0$. In that case, we first know by Lemma 3.3 that

$$\sum_{l=1}^{\infty} \mu^l c_l(t) \leq C(\delta, K_0, \mu)$$

for all $t \geq 0$, which implies by Lemma 3.5 that $c_1 \geq \delta_1$ for all t after a possible initial time layer. Then Theorem 2.1 directly follows from Proposition 3.7.

Assume now, that intervals (t_n^-, t_n^+) as above exist. Lemma 3.6 implies, since F is decreasing, that the sum of the lengths of those intervals is bounded, i.e.

$$t_* := \sum_n |t_n^+ - t_n^-| \leq C \frac{F(c(0))}{\delta} \leq \frac{CK_0}{\delta}.$$

Since $|\partial_t c_1| \leq \rho^2$ we also conclude that

$$|t_n^+ - t_n^-| \geq \frac{\delta}{C}.$$

Hence, the number N of intervals (t_n^-, t_n^+), is bounded as $N \leq \frac{CK_0}{\delta^2}$.
We now define a sequence μ_n, $n = 1, \dots, N$ in the following way:

$$\mu_0 = \mu$$

with μ as in (H6). Then μ_n is defined successively via

$$\mu_n = 1 + e^{-C(t_n^+ - t_n^-)}(\mu_{n-1} - 1).$$

With Lemma 3.2 we find

$$\sum_{l=1}^{\infty} \mu_n^l c_l(t_n^+) \leq \sum_{l=1}^{\infty} \mu_{n-1}^l c_l(t_n^-) + C(t_n^+ - t_n^-).$$

On the other hand Lemma 3.3 implies

$$\sum_{l=1}^{\infty} \mu_n^l c_l(t_{n+1}^-) \le \frac{C}{\delta(\mu_n - 1)} + \sum_{l=1}^{\infty} \mu_n^l c_l(t_n^+).$$

We find after N iterations and with

$$\bar{\mu} := \mu_N = 1 + e^{-Ct_*}(\mu - 1)$$

that

$$\sum_{l=1}^{\infty} (\bar{\mu})^l c_l(t_N) \le C(\delta, K_0, \mu).$$

For $t \ge t_N$ we have by definition $c_1 \le z_s - \delta/2$ and we find by Lemma 3.3 that

$$\sum_{l=1}^{\infty} (\bar{\mu})^l c_l(t) \le C(\delta, K_0, \mu)$$

for all $t \ge 0$. This implies with Lemma 3.5 that $c_1(t) \ge \delta_1$ for some $\delta_1 = \delta_1(\delta, K_0, \mu)$.
Thus, we can combine Lemma 3.6 and Proposition 3.7 to find the conclusion of Theorem 2.1.

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