WORKSHEET 5
WIENER’S INTEGRAL

In all the exercises, \((\Omega, \mathcal{A}, \mathbb{P})\) denotes the current probability space and \((B_t)_{t \geq 0}\) a (real) Brownian motion.

Exercise 1.

(1) Check that the random variable \(Y = \int_0^{+\infty} e^{-s} dB_s\) is well-defined.
(2) Give the law of \(Y\).

Exercise 2. Find two admissible functions \(f\) and \(g\) such that \(f \leq g\) and
\[
P\left[\int_0^1 f(s) dB_s > \int_0^1 g(s) dB_s\right] > 0.
\]

Exercise 3. Let \(f\) be an admissible function. Show that the process \((\int_0^t f(s) dB_s)_{t \geq 0}\) is a Gaussian process. Compute its mean and its covariance.

Exercise 4. Let \((X_t)_{t \geq 0}\) be given by:
\[
\forall t \geq 0, \quad X_t = \int_0^{t^{1/2}} (2s)^{1/2} dB_s.
\]
Show that \((X_t)\) is a Gaussian process. Compute its mean and its covariance. Deduce that \(X\) is a Brownian motion.

Exercise 5. Let \(V_0\) be a random variable independent of \(B\) and of Gaussian law \(\mathcal{N}(0, 1/2)\). We define the process \((V_t)_{t \geq 0}\) (so-called Ornstein-Uhlenbeck stationary process) by:
\[
\forall t \geq 0, \quad V_t = \exp(-t)V_0 + \int_0^t \exp(-(t-s)) dB_s.
\]
(1) Show that \((V_t)_{t \geq 0}\) is a Gaussian process.
(2) For any \(a > 0\), prove that \((V_{a+t})_{t \geq 0}\) and \((V_t)_{t \geq 0}\) have the same distribution.

Exercise 6. Let \(T > 0\). Show that
\[
\lim_{n \to +\infty} \mathbb{E}\left[\left(\sum_{i=1}^{n} (B_{T_i/n} - B_{T_{(i-1)/n}})^2 - T\right)^2\right] = 0.
\]

Exercise 7. Let \(T > 0\). Show that
\[
\int_0^T (1 + \frac{B_t}{n}) dB_t \xrightarrow{n \to \infty} \frac{1}{2} \int_0^T \exp(B_t) dB_t.
\]
Check first that the integrals are well-defined.
Exercise 8.  Let $T > 0$. For a given $n \geq 1$, we define the process

$$
\forall n \geq 0, \forall t \geq 0, \quad B^n_t = \sum_{i=0}^{n-1} B_{Ti/n} \mathbf{1}_{(Ti/n,T(i+1)/n]}(t).
$$

(1) Prove that $(B^n_t)_{t \geq 0}$ is a simple process w.r.t. the filtration generated by $B$.

(2) Show that

$$
\lim_{n \to +\infty} \mathbb{E} \int_0^T |B^n_t - B_t|^2 dt = 0.
$$

(3) What is the limit, in $L^2(\Omega)$, of

$$
\left( \int_0^T B^n_t \, dB_t \right)_{n \geq 1}.
$$

(4) Prove that

$$
B^2_T = 2 \int_0^T B^n_t \, dB_t + \sum_{i=1}^n (B_{Ti/n} - B_{T(i-1)/n})^2
$$

(5) By the previous exercise, deduce that

$$
B^2_T = 2 \int_0^T B_t \, dB_t + T.
$$